Black Hole Evolutions using Multiple Grid Patches

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Why Use Multiple Grid Patches?

Consider the numerical evolution of a (single) black hole spacetime

black hole ⇒ singularity ⇒ freezing slicing or excision
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- cubic (or other polyhedral) excision has causality problems at corners
- LEGO(TM) excision is messy (hard to finite difference stably/accurately)
- \((r, \theta, \phi)\) coordinates have severe \(z\) axis problems
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- well-posed outer BCs (maximally dissipative, constraint-preserving, . . .)
- matching Cauchy and characteristic codes
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Hence $r \times \{\text{multiple angular patches covering } S^2\}$ coordinates/grids
- $\Rightarrow$ smooth (spherical) inner and outer boundaries
- non-uniform radial coordinate $\Rightarrow$ easy (partial) fixed mesh refinement

Extension to multiple BHs possible with more general multiple-patch systems
6-Patch “Inflated-Cube” Coordinates and Grid

Paint $xyz$ grid lines on the faces of a cube, then inflate the cube into a sphere $\Rightarrow$ 6 angular patches on the sphere (neighborhoods of $\pm z$, $\pm x$, and $\pm y$ axes)
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Complicated to implement, but works well

[[movie of 4th order convergence in scalar field evolution]]
General Relativity

Non-scalar quantities ⇒ each patch uses its own local coordinate basis
(alternative: use global $xyz$ basis in all patches)

- write Einstein equations in 3-covariant form
- coordinate-transform $xyz$-basis initial data to each patch’s local basis
- coordinate-transform field variables when interpolating between neighboring patches (BSSN $\tilde{\Gamma}^k$ transformation needs dynamical $\tilde{g}^{ij}$)
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• coordinate-transform $xyz$-basis initial data to each patch’s local basis
• coordinate-transform field variables when interpolating between neighboring patches (BSSN $\tilde{\Gamma}^k$ transformation needs dynamical $\tilde{g}^{ij}$)
• must also write **gauge conditions** in 3-covariant form
  (Γ-driver is **not** 3-covariant ⇒ needs global basis)
Evolution of Kerr Spacetime

Initial Data: Kerr spacetime, spin \( a = 0.6 \), Kerr coordinates
(horizon is \( r = 1.8m \) sphere)

Grid: \( r_{\text{min}} = 1.5m, r_{\text{max}} = 248m \), 200 radial zones
\( \Delta \theta = 3^\circ, \Delta r = 0.10m \) (1.8m) at inner (outer) boundary

Finite Differencing: 4th order in space, RK4 time integration
5th order Lagrange polynomial interpatch interpolation
excision via 4th order Lagrange polynomial extrapolation
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BSSN evolution system
modified Bona-Massó lapse
time-independent (analytical) shift
(rotating) octant symmetry
standalone uniprocessor code, not in Cactus
Kerr Evolution: Convergence of Energy Constraint

Convergence is 4th order globally, 3rd order at interpatch boundaries
($t = 1000m$, $+z$ patch, $r =$ constant shell at $r = 2.19m$)

Convergence of $C$ at $t=1000m$
Angular Dependence in $+z$ patch at $w=0.12$ ($r=2.19m$)

$\sigma = \nu$

$\rho = \mu$

66k-wrmax4 $C/(1.5^4)$
50k-wrmax4 $C/(2.0^4)$
33k-wrmax4 $C/(3.0^4)$
Kerr Evolution: Excision and Outer Boundary Stability

Figure [movie] shows $45^\circ$ of $xz$ plane (logarithmic radial scale)
Evolution of Distorted Black Hole

Initial Data: Schwarzschild BH + Brill wave
Grid: $r_{\text{min}} = 0.76m$, $r_{\text{max}} = 48m$, 52 radial zones
   $\Delta \theta = 3.2^\circ$, $\Delta r = 0.05m$ (2.1$m$) at inner (outer) boundary
Finite Differencing: same as non-Cactus code (bugs!)
Cactus, Carpet (multi-patch and mesh-refinement driver)
Bona minimal-distortion–driver shift [joint work with Denis Pollney]
slice stretching limits evolution to $\sim 30m$
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Figure [movie] shows \( \text{xz plane} \) (linear radial scale)
General Relativistic Hydrodynamics

[Joint work with Ian Hawke (Southampton)]

- use high-resolution shock capturing (HRSC) schemes
- use interpatch interpolation schemes (e.g. ENO) that can handle shocks
- no problems with shocks crossing patch boundaries

[movie showing advection of discontinuous test-fluid profiles]
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  [movie showing advection of discontinuous test-fluid profiles]

Example: Relativistic Test-Fluid Accretion onto a Kerr BH \((a = 0.9)\) [test problem from Font, Ibanez and Papadopoulos (astro-ph/9810344)]

- Cactus, Carpet (multi-patch + mesh-refinement driver)
- Whisky (EU-network hydro code), PPM/Marquina solver (gr-qc/0501054)
- 5th order HRSC spatial finite differencing (weighted ENO)
- 4th order ENO interpatch interpolation
- RK4 time integration
- still relatively low resolution
  \((\Delta \theta = 4.5^\circ; \Delta r \approx 0.08m \text{ at BH, } \approx 1m \text{ at outer boundary})\)
Relativistic Test-Fluid Accretion onto a Kerr BH: Results

Figure shows contours of rest-mass density at $t = 200m$
needs more resolution $\Rightarrow$ need multiprocessor
Cactus/Carpet Infrastructure for Multipatch

Existing Thorns that Work with Multipatch

- “pointwise” initial data thorns
- other initial data thorns that do their own elliptic solve independent of the Cactus grid, then interpolate to the Cactus grid
- evolution thorns require only boundary-condition changes
- local diagnostics work unchanged
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Work in Progress

• multiprocessor
• global interpolation ⇒ apparent-horizon finding, wave extraction
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Future Infrastructure Plans

- add a Cartesian patch in the middle
  ⇒ can have initial data without a black hole
  ⇒ also gives main infrastructure for binary BH
- visualization: coordinate-transform non-scalar diagnostics to \(xyz\) basis
Binary Black Holes (Future)

Add Cartesian patches:
Conclusions

Advantages of Multiple Grid Patches

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- smooth outer boundary ⇒ easier to do well-posed outer BCs or Cauchy-characteristic matching
- non-uniform radial coordinate ⇒ easy (partial) fixed mesh refinement
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Disadvantages of Multiple Grid Patches

- implementation complexity (interpolation centering near corners & symmetry planes)
- diagnostics complexity (different tensor basis in each patch)
- mathematical complexity \( \Rightarrow \) hard to prove well-posedness, stability, etc.
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Plans

- better gauges: 3-covariant, avoid slice stretching, nice spatial $x^i$, ...
- Physics: proto-NS collapse, toroidal NS oscillations, Cauchy-characteristic matching, ..., binary BH
Born versus Einstein

Excerpts from discussion after Einstein’s Fall 1913 lecture in Vienna, “The present position of the problem of gravitation”:

Born: I should like to put to Herr Einstein a question, namely, how quickly the action of gravitation is propagated in your theory. That it happens with the speed of light does not elucidate it to me. There must be a very complicated connection between these ideas.

Einstein: It is extremely simple to write down the equations for the case when the perturbations that one introduces in the field are infinitely small. Then the $g$’s differ only infinitesimally from those that would be present without the perturbation. The perturbations then propagate with the same velocity as light.

Born: But for great perturbations things are surely very complicated?

Einstein: Yes, it is a mathematically complicated problem. It is especially difficult to find exact solutions of the equations, as the equations are nonlinear.