Constraint Preserving Boundary Treatment for the Einstein Equations in 2nd Order Form

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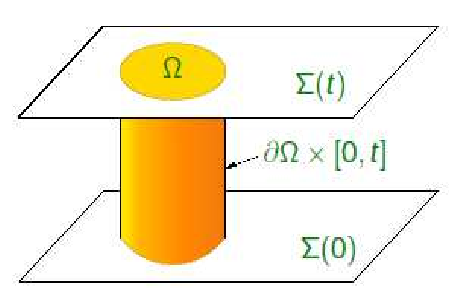
Outline

The Initial Boundary Value Problem

- To simulate spacetimes numerically on a finite grid we truncate the computational domain by introducing an artificial outer boundary.
- The boundary conditions should:
  - be compatible with the constraints
  - reduce reflections
  - yield a well-posed initial-boundary value problem.

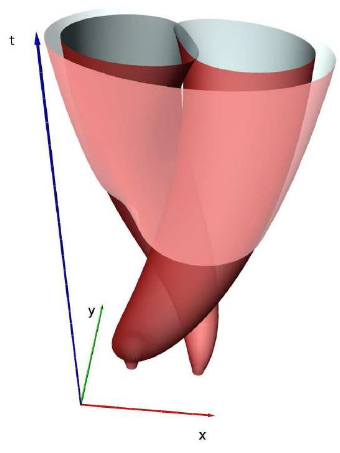


Past Work

- [*Stewart* 1998] Necessary conditions for well-posedness of linearized Einstein equations with constraint-preserving boundary conditions (Fourier-Laplace analysis)
- [*Friedrich* & *Nagy* 1999] To-date the only formulation proven to satisfy all the requirements for the *fully nonlinear* (vacuum) Einstein equations (frame formalism)
- [*Kreiss* & *Winicour* 2006] Well posed and constraint preserving boundary conditions for *linearized* Einstein Equations
- [*Buchman* & *Sarbach* 2006] Towards absorbing outer boundaries in General Relativity
- [*Rinne*, 2006] Stable radiation-controlling boundary conditions for the generalized harmonic Einstein equations

The AEIHarmonic Code

- Generalized harmonic system
- 2nd differential order in space
- Constraint damping
- 4th order finite differencing
- Moving lego-excision
- Mesh refinement (with Carpet)



Inspiral and Merger with Harmonic Coordinates. A smooth crossing of the horizons can clearly be seen.

"Generalised" Harmonic Coordinates

Coordinates:

- GH coordinates, x^μ, satisfy the condition $\Box x^\mu = \Gamma^\mu = F^\mu$.
- $F^\mu(g^{\alpha\beta}, x^\rho)$ as a source function chosen to fine tune gauge to address the requirements of specific simulations.
- Provides solutions of the EEs provided that the constraints:

$$C^\mu \equiv \Gamma^\mu - \widehat{\Gamma}^\mu = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\kappa} \left(\sqrt{-g} g^{\lambda\kappa}\right) - \widehat{\Gamma}^\mu = 0$$

and their time derivatives are initially satisfied.

Evolution Variables:

- We define the evolution variables $\tilde{g}^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu}$ and $Q^{\mu\nu} \equiv n^\rho \partial_\rho \tilde{g}^{\alpha\beta}$, where n^ρ is timelike.
- This simplifies the constraint equations to

$$C^\mu \equiv -\frac{1}{\sqrt{-g}} \partial_\alpha \tilde{g}^{\alpha\mu} - \widehat{\Gamma}^\mu$$

and gives us a first order in time evolution system.

Features of Generalized Harmonic Coordinates

- System of equations is manifestly symmetric hyperbolic (given reasonable metric conditions).
- Simplifies the evolution equations:
  - When the gradient of this condition is substituted for terms in Einstein equations, the PP of each metric element reduces to a simple wave equation:

$$g^{\gamma\delta} g_{\alpha\beta,\gamma\delta} + \ldots = 0$$

  - Constraints have the same form.
- The constraint equations may be incorporated into the generalized harmonic coordinate conditions.
- Gauge source terms for Harmonic coordinates allow free choice of gauge for Einstein equations.

Summation by Parts Boundaries

- A discrete difference operator is said to satisfy SBP for a scalar product $E = \langle u, v \rangle$ if the property $\langle u, Dv \rangle + \langle v, Du \rangle = (u \cdot v) \,|_a^b$ holds for all functions u, v in $[a, b]$.
- We can construct differencing which obeys SBP by finding stencils which obey:
  - Stencils D of a given order, τ, such that $Du = \frac{du}{dx} + \mathcal{O}(h^\tau)$,
  - determined up to the boundaries of the domain by solving the set of polynomials

$$Dx^m - \frac{dx^m}{dx} = 0, \qquad m = 0, 1, \ldots, \tau, \qquad (1)$$

  - which also obey

$$\langle u, Du \rangle = -\frac{1}{2} u^2(0) \qquad (2)$$

$$\langle u + v, D\,(u + v) \rangle_h = \langle D\,(u + v)\,, u + v \rangle_h - (u_0 + v_0)^2 \qquad (3)$$

- Second derivative operators may be obtained simply by repeated application of the 1st derivative operator, this results in a wide stencil. Instead we use the second derivative SBP operators described in [*Carpenter* 1994].
- One can construct a 3D SBP operator by applying the 1D operator to each direction. The resulting operator also satisfies SBP with respect to a diagonal scalar product

$$(u, v)_\Sigma = h_x h_y h_z \sum_{ijk} \sigma_{ijk} u_{ijk} \cdot v_{ijk},$$

Well-Posed Boundaries

- For well-posedness, the energy estimate $\xi^{(n)} = \| u(\cdot, t) \|^2$ of your system should satisfy $\| u(\cdot, t) \|^2 \leq K(t) \| u(\cdot, 0) \|^2$
- We use the SBP rule to derive an estimate for the time derivative of the energy of the system.
  - Integrate using the SBP rule
  - Substitute our boundary conditions
  - Applying that estimate as a penalty to our original equation
  - We can then choose coefficients for our boundary system which control the energy growth of the whole system.

$$
\begin{aligned}
\partial_t Q^{\mu\nu} &= -\frac{g^{it}}{g^{tt}} D_i^{(1)} Q^{\mu\nu} - (g^{ij} + \frac{g^{it} g^{jt}}{g^{tt}}) D_{ij}^{(2)} \tilde{g}^{\mu\nu} \\
&+ \frac{2g^{ij}}{g^{tt}\beta_0} H^{-1} E_{0_i} [(1 + \frac{g^{it}}{g^{tt}}) S_{i+} \tilde{g}^{\mu\nu} - \frac{Q^{\mu\nu}}{g^{tt}} + \frac{2x}{r^2} (\tilde{g}^{\mu\nu} - \tilde{g}_0^{\mu\nu})] \\
&+ \frac{2g^{ij}}{g^{tt}\beta_N} H^{-1} E_{N_i} [(1 - \frac{g^{it}}{g^{tt}}) S_{i-} \tilde{g}^{\mu\nu} + \frac{Q^{\mu\nu}}{g^{tt}} + \frac{2x}{r^2} (\tilde{g}^{\mu\nu} - \tilde{g}_N^{\mu\nu})]
\end{aligned}
$$

Well-Posed Boundaries

- For well-posedness, the energy estimate $\xi^{(n)} = \| u(\cdot, t) \|^2$ of your system should satisfy $\| u(\cdot, t) \|^2 \leq K(t) \| u(\cdot, 0) \|^2$
- We use the SBP rule to derive an estimate for the time derivative of the energy of the system.
  - Integrate using the SBP rule.
    For the wave equation with shift:

$$u_{tt} = (\frac{-g^{ij}}{g^{tt}} \partial_i \partial_j - 2\frac{g^{it}}{g^{tt}} \partial_i \partial_t) u. \qquad (4)$$

The time derivative of the energy:

$$\begin{aligned} \frac{d}{dt}\mathcal{E} &= \frac{d}{dt}\left(\|u_t\|^2 + \|-\frac{g^{ij}}{g^{tt}} u_i u_j\| \right) \\ &= (\langle u_t, u_{tt}\rangle + \langle u_{tt}, u_t\rangle) - \frac{g^{ij}}{g^{tt}} (\langle u_i, u_{jt}\rangle + \langle u_{it}, u_j\rangle) . \end{aligned} \qquad (5)$$

integrating by SBP

$$\frac{d}{dt}\mathcal{E} = -2[\frac{g^{ij}}{g^{tt}}(u_t u_j)\,|_{x_i=0}^{x_i=N_i} + \frac{g^{it}}{g^{tt}}(u_t^2)\,|_{x_i=0}^{x_i=N_i}] = 0\,. \qquad (6)$$

Well-Posed Boundaries

- For well-posedness, the energy estimate $\xi^{(n)} = \| u(\cdot, t) \|^2$ of your system should satisfy $\| u(\cdot, t) \|^2 \leq K(t) \| u(\cdot, 0) \|^2$
- We use the SBP rule to derive an estimate for the time derivative of the energy of the system.
  - Integrate using the SBP rule
  - Substitute our boundary conditions:
    the change in energy is determined by fluxes at the boundary points, $x_i = 0$ and $x_i = N_i$.

$$[\beta_{x_i=0}\partial_t + \alpha_{x_i=0}\partial_i + \delta_{x_i=0}](u - u_0) = 0 \quad (7)$$
$$[\beta_{x_i=N}\partial_t - \alpha_{x_i=N}\partial_i - \delta_{x_i=N}](u - u_0) = 0 \quad (8)$$

into the estimate giving:

$$\frac{d}{dt}\mathcal{E} = -2[(\frac{\alpha_{N_i}}{\beta_{N_i}}u_t^2 - \frac{g^{it}}{g^{tt}}u_t^2)\mid_{x_i=N_i} - (\frac{\alpha_{0_i}}{\beta_{0_i}}u_t^2 - \frac{g^{it}}{g^{tt}}u_t^2)\mid_{x_i=0}] \quad (9)$$

Well-Posed Boundaries

- For well-posedness, the energy estimate $\xi^{(n)} = \|u(\cdot,t)\|^2$ of your system should satisfy $\|u(\cdot,t)\|^2 \leq K(t)\|u(\cdot,0)\|^2$
- We use the SBP rule to derive an estimate for the time derivative of the energy of the system.
  - Integrate using the SBP rule
  - Substitute our boundary conditions
  - Applying that estimate as a penalty to our original equation:

$$\begin{aligned} u_{tt} &= -\frac{g^{ij}}{g^{tt}} D_{ij}^{(2)} u - 2\frac{g^{it}}{g^{tt}} D_i^{(1)} u_t \\ &+\tau_{0_i} H^{-1} E_{0_i}(\alpha_{0_i} u_t + \beta_{0_i} S_i u + \delta_{0_i} u) \\ &+\tau_{N_i} H^{-1} E_{N_i}(\alpha_{N_i} u_t + \beta_{N_i} S_i u + \delta_{N_i} u)\,. \end{aligned} \tag{10}$$

gives an estimate:

$$\begin{aligned} \frac{d}{dt}\mathcal{E} &= (\tau_{N_i}\alpha_{N_i} - \frac{g^{it}}{g^{tt}}) u_t^\top E_{N_i} u_t + 2(\tau_{0_i}\alpha_{0_i} + \frac{g^{it}}{g^{tt}}) u_t^\top E_{0_i} u_t \\ &+2(\tau_{N_i}\beta_{N_i} - \frac{g^{ij}}{g^{tt}}) u_t^\top E_{N_i} S_i u + 2(\tau_{0_i}\beta_{0_i} + \frac{g^{ij}}{g^{tt}}) u_t^\top E_{0_i} S_i u\,. \end{aligned}$$

Well-Posed Boundaries

- For well-posedness, the energy estimate $\xi^{(n)} = \| u(\cdot, t) \|^2$ of your system should satisfy $\| u(\cdot, t) \|^2 \leq K(t) \| u(\cdot, 0) \|^2$
- We use the SBP rule to derive an estimate for the time derivative of the energy of the system.
  - Integrate using the SBP rule
  - Substitute our boundary conditions
  - Applying that estimate as a penalty to our original equation
  - We can then choose coefficients for our boundary system which control the energy growth of the whole system.

    The free parameters τ_0 and τ_N can be used to eliminatethe $u_t^\top E_{N_i} S_i u$ terms, by setting $\tau_0 \beta_0 = -\gamma^{ij}/\gamma^{tt}$ and $\tau_N \beta_N = \gamma^{ij}/\gamma^{tt}$. Then, the energy evolves according to

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E} &= -2(\beta_{N_i}\frac{g^{it}}{g^{tt}} - \alpha_{N_i}\frac{g^{ij}}{g^{tt}})\beta_{N_i}^{-1} u_t^\top E_{N_i} u_t \\
&\quad +2(\beta_{0_i}\frac{g^{it}}{g^{tt}} - \alpha_{0_i}\frac{g^{ij}}{g^{tt}})\beta_{0_i}^{-1} u_t^\top E_{0_i} u_t = 0. \qquad (11)
\end{aligned}
$$

Well-Posed Boundaries

- For well-posedness, the energy estimate $\xi^{(n)} = \|u(\cdot,t)\|^2$ of your system should satisfy $\|u(\cdot,t)\|^2 \leq K(t)\|u(\cdot,0)\|^2$
- We use the SBP rule to derive an estimate for the time derivative of the energy of the system.
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  - Substitute our boundary conditions
  - Applying that estimate as a penalty to our original equation
  - We can then choose coefficients for our boundary system which control the energy growth of the whole system.

    The resulting semi-discrete evolution equation is given by

$$\begin{aligned}
\partial_t Q^{\mu\nu} &= -\frac{g^{it}}{g^{tt}} D_i^{(1)} Q^{\mu\nu} - (g^{ij} + \frac{g^{it} g^{jt}}{g^{tt}}) D_{ij}^{(2)} \tilde{g}^{\mu\nu} \\
&+ \frac{2g^{ij}}{g^{tt}\beta_0} H^{-1} E_{0_i} [(1 + \frac{g^{it}}{g^{tt}}) S_{i+} \tilde{g}^{\mu\nu} - \frac{Q^{\mu\nu}}{g^{tt}} + \frac{2x}{r^2} (\tilde{g}^{\mu\nu} - \tilde{g}_0^{\mu\nu})] \\
&+ \frac{2g^{ij}}{g^{tt}\beta_N} H^{-1} E_{N_i} [(1 - \frac{g^{it}}{g^{tt}}) S_{i-} \tilde{g}^{\mu\nu} + \frac{Q^{\mu\nu}}{g^{tt}} + \frac{2x}{r^2} (\tilde{g}^{\mu\nu} - \tilde{g}_N^{\mu\nu})]
\end{aligned}$$

Constraint Preservation

- Sommerfeld-type outgoing conditions:

$$\left(\partial_t - \partial_x - \frac{1}{r}\right)\left(\gamma^{\mu\nu} - \gamma_0^{\mu\nu}\right) = 0$$

- For CP Boundaries we set the four $\gamma^{t\mu}$ from the constraints:

$$C^\mu = -\partial_t \gamma^{t\mu} - \partial_i \gamma^{i\mu} - F^\mu = 0$$

  and we derive a set of outgoing conditions which specify the other 6 metric components:

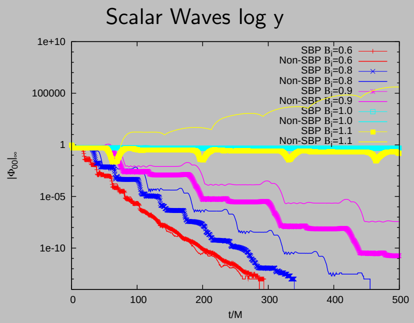
$$\left(\partial_x + \partial_t + \frac{1}{r}\right)\left(\gamma^{AB} - \gamma_0^{AB}\right) = 0$$

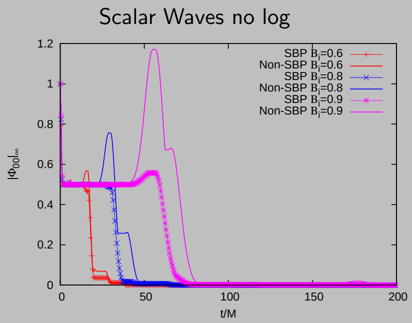
$$\left(\partial_x + \partial_t + \frac{1}{r}\right)\left(\gamma^{tA} - \gamma^{xA} - \gamma_0^{tA} + \gamma_0^{xA}\right) = 0$$

$$\left(\partial_x + \partial_t + \frac{1}{r}\right)\left(\gamma^{tt} - 2\gamma^{xt} + \gamma^{xx} - \gamma_0^{tt} + 2\gamma_0^{xt} - \gamma_0^{xx}\right) = 0$$

see: [2] {Kreiss and Winicour, gr-qc 0602051}

Results for High Shifts



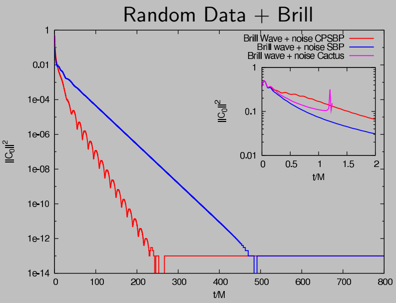


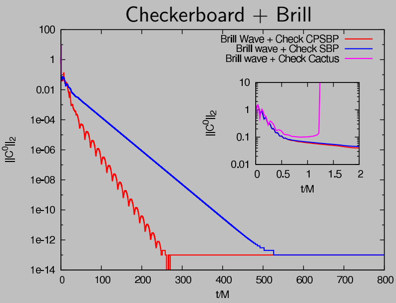
- Tests with Scalarwave testbed
- Stability test for various shifts $(0.6 < \frac{\gamma^{it}}{\gamma^{tt}} < 1.1)$:

$$u_{tt} = -2\frac{\gamma^{it}}{\gamma^{tt}}u_{it} - \frac{\gamma^{ij}}{\gamma^{tt}}u_{ij}$$

- Thin = Standard Somerfeld, Thick = SBP
- Reflections for standard BCs clearly visible in right hand plot

Robust Stability Tests

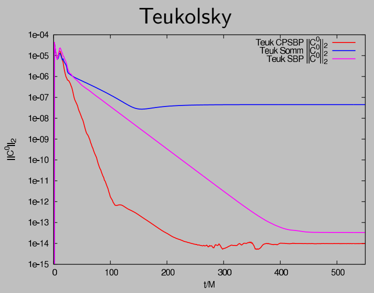


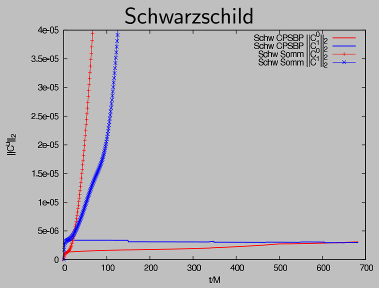


- Random Data + Brill Wave
  - Random Kernel Amplitude = 0.1
  - Brill Wave Amplitude = 0.5
  - dx = 0.2 xmax = 7.1
- Runs stable for in nonlinear regime for Brill Waves.
- Stable for random data
- Standard Sommerfeld type breaks rapidly for this simulation

- Checkerboard Data + Brill Wave
  - for each $x(i), y(j), z(k)$ we add $(-1)^{i+j+k}A$ highest frequency noise possible
  - Checker Kernel $A = \pm 0.2$
  - Brill Wave Amplitude = 0.5
  - dx = 0.2 xmax = 7.1
- Standard sommerfeld seen in green (breaks quickly)

Results for Teukolsky/Brill Wave and Schwarzschild Runs

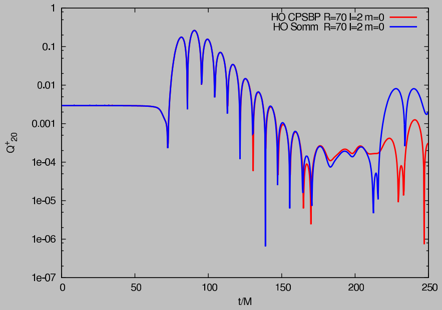


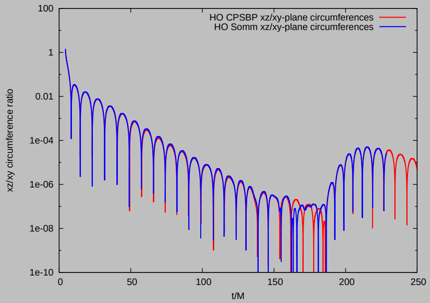


- High Amplitude Teukolsky Waves
- Constraint Norms for runs with:
  - Constraint Preserving 'SBP' = Red
  - Pure SBP = Magenta
  - Standard sommerfeld-type = Blue

- Schwarzschild run with boundaries too close in (40 M) for sommerfeld-type boundaries
- CPSBP remains stable

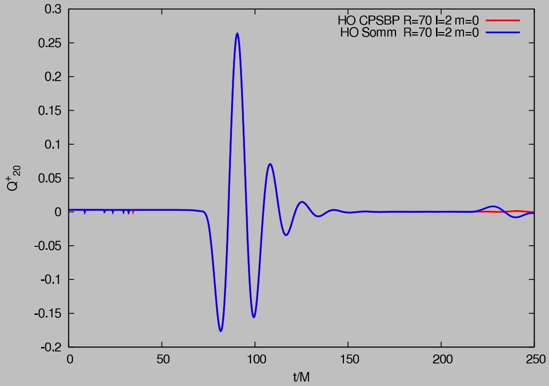
Head-on Runs with CPSBP





- Headon Collison (mass 0.5, 2.5 M separation)
- L2 Norm of Constraints for CPSBP vs regular boundaries
- Significant improvement in constraint preservation
- Circumference ratios almost identical
- Some boundary effects are visible for the standard BC runs which are not in the CPSBP run

Conclusions



- SBP provides a provably well-posed and demonstrably stable IBVP for Generalized Harmonic evolutions on a Cartesian grid
- Stands up to stability tests
- We have developed a method which allows us to consistently use SBP on a Cartesian grid for corners and edges, and for a 2nd order in space system

Thank You.

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Comm. Math. Phys., 201:619-655, 15 Sept. 1999.

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Well-posed Initial-boundary Evolution in General Relativity.
Phys. Rev. D, 68:041501(1)041501(5), 2003.

Constraint Damping

- The constraint equations are the generalized harmonic coordinate conditions: $C^\mu \equiv \Gamma^\mu - \widehat{\Gamma}^\mu = 0$
- constraint adjustment is done by the term

$$A^{\mu\nu} = C^\rho A_\rho^{\mu\nu} \left(x^\alpha, g_{\alpha\beta}, \partial_\gamma g_{\alpha\beta}\right)$$

  in the evolution equations

$$\partial_\alpha \left(g^{\alpha\beta} \partial_\beta \tilde{g}^{\mu\nu}\right) + S^{\mu\nu} \left(g, \partial g\right) + \sqrt{-g} A^{\mu\nu}$$

$$+2\sqrt{-g} \nabla^{(\mu} F^{\nu)} - \tilde{g}^{\mu\nu} \nabla_\alpha F^\alpha = 0.$$

- Dissipation: $\dot{f} \longrightarrow \dot{f} + \epsilon(\delta^{ij} D_{+i} D_{-i}) w(\delta^{ij} D_{+i} D_{-i}) f$ where w is a weight factor that vanishes at the outer boundary. With $D_{+i} D_{-i}$ from blended SBP stencils.

HarmonicExcision

$\left(n^i D_{+i}\right)^3 \dot{f} = 0$ to all guard points, in layers stratified by length of the outward normal pointing vector, from out to in.

LegoExcision with excision coefficients $\frac{x^\mu}{r}$ extrapolated around a smooth virtual surface for the inner boundary.

Radiation outer boundary conditions (i.e. outgoing only).