Constraint Preserving Boundary Treatment
for the Einstein Equations in 2nd Order Form

Jennifer Seiler

Collaborators:
B Szilagyi, D Pollney, L Rezzolla

Max-Planck-Institute for Gravitational Physics
Potsdam, Germany

SFB/TR7 Videoseminar
AEI Golm, Germany
20th January 2008
Outline

1 Introduction
   - The Initial Boundary Value Problem

2 Past Work

3 The AEIHarmonic Code

4 Overview of the Harmonic Coordinates
   - Description
   - Features

5 Summation By Parts

6 Constraint Preservation

7 Results
   - High Shifts
   - Robust Stability Tests
   - Teukolsky
   - Headon

8 Conclusions
The Initial Boundary Value Problem

- To simulate spacetimes numerically on a finite grid we truncate the computational domain by introducing an artificial outer boundary.
- The boundary conditions should:
  - be compatible with the constraints
  - reduce reflections
  - yield a well-posed initial-boundary value problem.
Past Work

- **[Stewart 1998]** Necessary conditions for well-posedness of linearized Einstein equations with constraint-preserving boundary conditions (Fourier-Laplace analysis)
- **[Friedrich & Nagy 1999]** To-date the only formulation proven to satisfy all the requirements for the fully nonlinear (vacuum) Einstein equations (frame formalism)
- **[Kreiss & Winicour 2006]** Well posed and constraint preserving boundary conditions for linearized Einstein Equations
- **[Buchman & Sarbach 2006]** Towards absorbing outer boundaries in General Relativity
- **[Rinne, 2006]** Stable radiation-controlling boundary conditions for the generalized harmonic Einstein equations
The AEIHarmoinc Code

- Generalized harmonic system
- 2nd differential order in space
- Constraint damping
- 4th order finite differencing
- Moving lego-excision
- Mesh refinement (with Carpet)

Inspiral and Merger with Harmonic Coordinates. A smooth crossing of the horizons can clearly be seen.
"Generalised" Harmonic Coordinates

Coordinates:

- GH coordinates, $x^\mu$, satisfy the condition $\Box x^\mu = \Gamma^\mu = F^\mu$.
- $F^\mu(g^{\alpha\beta}, x^\rho)$ as a source function chosen to finely tune gauge to address the requirements of specific simulations.
- Provides solutions of the EEs provided that the constraints:

$$C^\mu \equiv \Gamma^\mu - \tilde{\Gamma}^\mu = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\kappa} \left( \sqrt{-g} g^{\lambda\kappa} \right) - \tilde{\Gamma}^\mu = 0$$

and their time derivatives are initially satisfied.

Evolution Variables:

- We define the evolution variables $\tilde{g}^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu}$ and $Q^{\mu\nu} \equiv n^\rho \partial_\rho \tilde{g}^{\alpha\beta}$, where $n^\rho$ is timelike.
- This simplifies the constraint equations to

$$C^\mu \equiv -\frac{1}{\sqrt{-g}} \partial_\alpha \tilde{g}^{\alpha\mu} - \tilde{\Gamma}^\mu$$

and gives us a first order in time evolution system.
Features of Generalized Harmonic Coordinates

- System of equations is manifestly symmetric hyperbolic (given reasonable metric conditions).
- Simplifies the evolution equations:
  - When the gradient of this condition is substituted for terms in Einstein equations, the PP of each metric element reduces to a simple wave equation:
    \[ g^{\gamma\delta} g_{\alpha\beta,\gamma\delta} + \ldots = 0 \]
- Constraints have the same form.
- The constraint equations may be incorporated into the generalized harmonic coordinate conditions.
- Gauge source terms for Harmonic coordinates allow free choice of gauge for Einstein equations.
Summation by Parts Boundaries

- A discrete difference operator is said to satisfy SBP for a scalar product $E = \langle u, v \rangle$ if the property $\langle u, Dv \rangle + \langle v, Du \rangle = (u \cdot v) \big|_a^b$ holds for all functions $u, v$ in $[a, b]$.
- We can construct differencing which obeys SBP by finding stencils which obey:
  - Stencils $D$ of a given order, $\tau$, such that $Du = \frac{du}{dx} + O(h^\tau)$, determined up to the boundaries of the domain by solving the set of polynomials
    
    \begin{equation}
    Dx^m - \frac{dx^m}{dx} = 0, \quad m = 0, 1, \ldots, \tau, \tag{1}
    \end{equation}
  
  - which also obey
    
    \begin{equation}
    \langle u, Du \rangle = -\frac{1}{2}u^2(0) \tag{2}
    \end{equation}
    
    \begin{equation}
    \langle u + v, D(u + v) \rangle_h = \langle D(u + v), u + v \rangle_h - (u_0 + v_0)^2 \tag{3}
    \end{equation}

- Second derivative operators may be obtained simply by repeated application of the 1st derivative operator, this results in a wide stencil. Instead we use the second derivative SBP operators described in [Carpenter 1994].
- One can construct a 3D SBP operator by applying the 1D operator to each direction. The resulting operator also satisfies SBP with respect to a diagonal scalar product
  
  \begin{equation}
  (u, v)_\Sigma = h_x h_y h_z \sum_{ijk} \sigma_{ijk} u_{ijk} \cdot v_{ijk}, \tag{4}
  \end{equation}
Well-Posed Boundaries

- For well-posedness, the energy estimate $\xi^{(n)} = \|u(\cdot, t)\|^2$ of your system should satisfy $\|u(\cdot, t)\|^2 \leq K(t) \|u(\cdot, 0)\|^2$

- We use the SBP rule to derive an estimate for the time derivative of the energy of the system.
  - Integrate using the SBP rule
  - Substitute our boundary conditions
  - Applying that estimate as a penalty to our original equation
  - We can then choose coefficients for our boundary system which control the energy growth of the whole system.

\[
\partial_t Q^{\mu\nu} = -\frac{g^{it}}{g^{tt}} D_i^{(1)} Q^{\mu\nu} - (g^{ij} + \frac{g^{it} g^{jt}}{g^{tt}}) D_{ij}^{(2)} \tilde{g}^{\mu\nu} \\
+ \frac{2g^{ij}}{g^{tt} \beta_0} H^{-1} E_{0i} [(1 + \frac{g^{it}}{g^{tt}}) S_i + \tilde{g}^{\mu\nu} - \frac{Q^{\mu\nu}}{g^{tt}} + \frac{2x}{r^2} (\tilde{g}^{\mu\nu} - \tilde{g}_0^{\mu\nu})] \\
+ \frac{2g^{ij}}{g^{tt} \beta_N} H^{-1} E_{Ni} [(1 - \frac{g^{it}}{g^{tt}}) S_i - \tilde{g}^{\mu\nu} + \frac{Q^{\mu\nu}}{g^{tt}} + \frac{2x}{r^2} (\tilde{g}^{\mu\nu} - \tilde{g}_N^{\mu\nu})]
\]
Well-Posed Boundaries

- For well-posedness, the energy estimate $\xi^{(n)} = \|u(\cdot, t)\|^2$ of your system should satisfy $\|u(\cdot, t)\|^2 \leq K(t) \|u(\cdot, 0)\|^2$.
- We use the SBP rule to derive an estimate for the time derivative of the energy of the system.
  - Integrate using the SBP rule.
    For the wave equation with shift:
    
    $$u_{tt} = \left(-\frac{g^{ij}}{g_{tt}} \partial_{i} \partial_{j} - 2\frac{g^{it}}{g_{tt}} \partial_{i} \partial_{t}\right)u.$$  

    \hspace{1cm} (4)

    The time derivative of the energy:
    
    $$\frac{d}{dt} E = \frac{d}{dt} \left(\|u_t\|^2 + \| -\frac{g^{ij}}{g_{tt}} u_i u_j \| \right)$$

    $$= \left( \langle u_t, u_{tt} \rangle + \langle u_{tt}, u_t \rangle \right) - \frac{g^{ij}}{g_{tt}} (\langle u_i, u_{jt} \rangle + \langle u_{it}, u_j \rangle).$$  

    \hspace{1cm} (5)

    integrating by SBP

    $$\frac{d}{dt} E = -2\left[ \frac{g^{ij}}{g_{tt}} (u_t u_j) \big|_{x_j=N_i}^{x_j=0} + \frac{g^{it}}{g_{tt}} (u_t^2) \big|_{x_j=N_i}^{x_j=0} \right] = 0.$$  

    \hspace{1cm} (6)
Well-Posed Boundaries

- For well-posedness, the energy estimate $\xi^{(n)} = \| u (\cdot, t) \|^2$ of your system should satisfy $\| u (\cdot, t) \|^2 \leq K (t) \| u (\cdot, 0) \|^2$.

- We use the SBP rule to derive an estimate for the time derivative of the energy of the system.
  - Integrate using the SBP rule
  - Substitute our boundary conditions:
    - the change in energy is determined by fluxes at the boundary points, $x_i = 0$ and $x_i = N_i$.

\[
\begin{align*}
[\beta_{x_i=0} \partial_t + \alpha_{x_i=0} \partial_i + \delta_{x_i=0}] (u - u_0) & = 0 \quad (7) \\
[\beta_{x_i=N} \partial_t - \alpha_{x_i=N} \partial_i - \delta_{x_i=N}] (u - u_0) & = 0 \quad (8)
\end{align*}
\]

- into the estimate giving:

\[
\frac{d}{dt} E = -2 \left[ \left( \frac{\alpha_{N_i}}{\beta_{N_i}} u_t^2 - \frac{g_{tt}}{g_{tt}} u_t^2 \right) \bigg|_{x_i=N_i} - \left( \frac{\alpha_{0_i}}{\beta_{0_i}} u_t^2 - \frac{g_{tt}}{g_{tt}} u_t^2 \right) \bigg|_{x_i=0} \right] \quad (9)
\]
For well-posedness, the energy estimate \( \xi^{(n)} = \| u (\cdot, t) \|^{2} \) of your system should satisfy \( \| u (\cdot, t) \|^{2} \leq K(t) \| u (\cdot, 0) \|^{2} \).

We use the SBP rule to derive an estimate for the time derivative of the energy of the system.

- Integrate using the SBP rule
- Substitute our boundary conditions
- Applying that estimate as a penalty to our original equation:

\[
\begin{align*}
    u_{tt} & = -\frac{g_{ij}}{g_{tt}} D_{ij}^{(2)} u - 2 \frac{g_{it}}{g_{tt}} D_{i}^{(1)} u_t \\
            & \quad + \tau_{0_i} H^{-1} E_{0_i} (\alpha_{0_i} u_t + \beta_{0_i} S_i u + \delta_{0_i} u) \\
            & \quad + \tau_{N_i} H^{-1} E_{N_i} (\alpha_{N_i} u_t + \beta_{N_i} S_i u + \delta_{N_i} u). \\
\end{align*}
\]  

(10)

gives an estimate:

\[
\begin{align*}
    \frac{d}{dt} \mathcal{E} & = (\tau_{N_i} \alpha_{N_i} - \frac{g_{it}}{g_{tt}}) u_t^T E_{N_i} u_t + 2(\tau_{0_i} \alpha_{0_i} + \frac{g_{it}}{g_{tt}}) u_t^T E_{0_i} u_t \\
            & \quad + 2(\tau_{N_i} \beta_{N_i} - \frac{g_{ij}}{g_{tt}}) u_t^T E_{N_i} S_i u + 2(\tau_{0_i} \beta_{0_i} + \frac{g_{ij}}{g_{tt}}) u_t^T E_{0_i} S_i u.
\end{align*}
\]
Well-Posed Boundaries

- For well-posedness, the energy estimate $\xi^{(n)} = \|u(\cdot, t)\|^2$ of your system should satisfy $\|u(\cdot, t)\|^2 \leq K(t)\|u(\cdot, 0)\|^2$.
- We use the SBP rule to derive an estimate for the time derivative of the energy of the system.
  - Integrate using the SBP rule
  - Substitute our boundary conditions
  - Applying that estimate as a penalty to our original equation
  - We can then choose coefficients for our boundary system which control the energy growth of the whole system.

The free parameters $\tau_0$ and $\tau_N$ can be used to eliminate the $u_t^\top E_{Ni} S_i u$ terms, by setting $\tau_0 \beta_0 = -\gamma^{ij}/\gamma^{tt}$ and $\tau_N \beta_N = \gamma^{ij}/\gamma^{tt}$. Then, the energy evolves according to

$$
\frac{d}{dt} \mathcal{E} = -2(\beta_{Ni} \frac{g^{it}}{g^{tt}} - \alpha_{Ni} \frac{g^{ij}}{g^{tt}}) \beta_{Ni}^{-1} u_t^\top E_{Ni} u_t
$$

$$
+2(\beta_{0i} \frac{g^{it}}{g^{tt}} - \alpha_{0i} \frac{g^{ij}}{g^{tt}}) \beta_{0i}^{-1} u_t^\top E_{0i} u_t = 0. \quad (11)
$$
Well-Posed Boundaries

- For well-posedness, the energy estimate \( \xi^{(n)} = \| u(\cdot, t) \|^{2} \) of your system should satisfy \( \| u(\cdot, t) \|^{2} \leq K(t) \| u(\cdot, 0) \|^{2} \)

- We use the SBP rule to derive an estimate for the time derivative of the energy of the system.
  - Integrate using the SBP rule
  - Substitute our boundary conditions
  - Applying that estimate as a penalty to our original equation
  - We can then choose coefficients for our boundary system which control the energy growth of the whole system.

The resulting semi-discrete evolution equation is given by

\[
\partial_{t} Q^{\mu \nu} = -\frac{g^{it}}{g^{tt}} D_{i}^{(1)} Q^{\mu \nu} - (g^{ij} + \frac{g^{it} g^{jt}}{g^{tt}}) D_{ij}^{(2)} \tilde{g}^{\mu \nu}
\]

\[
+ \frac{2g^{ij}}{g^{tt} \beta_{0}} H^{-1} E_{0i} [(1 + \frac{g^{it}}{g^{tt}}) S_{i} + \tilde{g}^{\mu \nu} - \frac{Q^{\mu \nu}}{g^{tt}} + \frac{2x}{r^{2}} (\tilde{g}^{\mu \nu} - \tilde{g}_{0}^{\mu \nu})]
\]

\[
+ \frac{2g^{ij}}{g^{tt} \beta_{N}} H^{-1} E_{Ni} [(1 - \frac{g^{it}}{g^{tt}}) S_{i} - \tilde{g}^{\mu \nu} + \frac{Q^{\mu \nu}}{g^{tt}} + \frac{2x}{r^{2}} (\tilde{g}^{\mu \nu} - \tilde{g}_{N}^{\mu \nu})]
\]
Constraint Preservation

- Sommerfeld-type outgoing conditions:
  \[
  \left( \partial_t - \partial_x - \frac{1}{r} \right) (\gamma^{\mu\nu} - \gamma_0^{\mu\nu}) = 0
  \]

- For CP Boundaries we set the four \( \gamma^{t\mu} \) from the constraints:
  \[
  C^\mu = -\partial_t \gamma^{t\mu} - \partial_i \gamma^{i\mu} - F^\mu = 0
  \]
  and we derive a set of outgoing conditions which specify the other 6 metric components:
  \[
  \left( \partial_x + \partial_t + \frac{1}{r} \right) (\gamma^{AB} - \gamma_0^{AB}) = 0
  \]
  \[
  \left( \partial_x + \partial_t + \frac{1}{r} \right) (\gamma^{tA} - \gamma^{xA} - \gamma_0^{tA} + \gamma_0^{xA}) = 0
  \]
  \[
  \left( \partial_x + \partial_t + \frac{1}{r} \right) (\gamma^{tt} - 2\gamma^{xt} + \gamma^{xx} - \gamma_0^{tt} + 2\gamma_0^{xt} - \gamma_0^{xx}) = 0
  \]

see: [2] {Kreiss and Winicour, gr-qc 0602051}
Results for High Shifts

- Tests with Scalarwave testbed
- Stability test for various shifts $(0.6 < \frac{\gamma^{it}}{\gamma^{tt}} < 1.1)$:

  \[ u_{tt} = -2 \frac{\gamma^{it}}{\gamma^{tt}} u_{it} - \frac{\gamma^{ij}}{\gamma^{tt}} u_{ij} \]

- Thin = Standard Sommerfeld, Thick = SBP
- Reflections for standard BCs clearly visible in right hand plot
Robust Stability Tests

Random Data + Brill Wave
- Random Kernel Amplitude = 0.1
- Brill Wave Amplitude = 0.5
- $dx = 0.2$, $xmax = 7.1$
- Runs stable for nonlinear regime for Brill Waves.
- Stable for random data
- Standard Sommerfeld type breaks rapidly for this simulation

Checkerboard Data + Brill Wave
- for each $x(i), y(j), z(k)$ we add $(-1)^{i+j+k}$ A highest frequency noise possible
- Checker Kernel $A = \pm 0.2$
- Brill Wave Amplitude = 0.5
- $dx = 0.2$, $xmax = 7.1$
- Standard Sommerfeld seen in green (breaks quickly)
Results for Teukolsky/Brill Wave and Schwarzschild Runs

High Amplitude Teukolsky Waves
Constraint Norms for runs with:
- Constraint Preserving 'SBP' = Red
- Pure SBP = Magenta
- Standard sommerfeld-type = Blue

Schwarzschild run with boundaries too close in (40 M) for sommerfeld-type boundaries
CPSBP remains stable
Head-on Runs with CPSBP

- Headon Collison (mass 0.5, 2.5 M separation)
- L2 Norm of Constraints for CPSBP vs regular boundaries
- Significant improvement in constraint preservation

Circumference ratios almost identical

Some boundary effects are visible for the standard BC runs which are not in the CPSBP run
Conclusions

- SBP provides a provably well-posed and demonstrably stable IBVP for Generalized Harmonic evolutions on a Cartesian grid
- Stands up to stability tests
- We have developed a method which allows us to consistently use SBP on a Cartesian grid for corners and edges, and for a 2nd order in space system
Thank You.
[G. Calabrese and C. Gundlach, gr-qc 0509119]
Discrete Boundary Treatment for the Shifted Wave Equation in Second Order Form and Related Problems.

[Kreiss and Winicour, gr-qc 0602051]
Problems Which are Well-Posed in a Generalised Sense With Applications to the Einstein Equations.

[G. Calabrese, J. Pullin, O. Reula, O. Sarbach, and M. Tiglio]
Well Posed Constraint-preserving Boundary Conditions For the Linearized Einstein Equations.

Initial Boundary Value Problem for Einstein’s Vacuum Field Equation.

[B. Szilagyi and J. Winicour, PRD 68:041501]
Well-posed Initial-boundary Evolution in General Relativity.
Constraint Damping

- The constraint equations are the generalized harmonic coordinate conditions: \( C^\mu \equiv \Gamma^\mu - \hat{\Gamma}^\mu = 0 \)
- Constraint adjustment is done by the term
  \[
  A^{\mu\nu} = C^\rho A^{\mu\nu}_\rho (x^\alpha, g_{\alpha\beta}, \partial_\gamma g_{\alpha\beta})
  \]
  in the evolution equations
  \[
  \partial_\alpha (g^{\alpha\beta} \partial_\beta \tilde{g}^{\mu\nu}) + S^{\mu\nu} (g, \partial g) + \sqrt{-g} A^{\mu\nu}
  \]
  \[
  + 2 \sqrt{-g} \nabla^{(\mu} F^{\nu)} - \tilde{g}^{\mu\nu} \nabla_\alpha F^\alpha = 0.
  \]
- Dissipation: \( \dot{f} \rightarrow \dot{f} + \epsilon (\delta^{ij}D_{+i}D_{-i}) w (\delta^{ij}D_{+i}D_{-i}) f \) where \( w \) is a weight factor that vanishes at the outer boundary. With \( D_{+i}D_{-i} \) from blended SBP stencils.
\[(n^i D_{+i})^3 \dot{f} = 0 \text{ to all guard points, in layers stratified by length of the outward normal pointing vector, from out to in.}\]

LegoExcision with excision coefficients \[\frac{\chi^\mu}{r}\] extrapolated around a smooth virtual surface for the inner boundary.

Radiation outer boundary conditions (i.e. outgoing only).