# Nonlinear spherical sound waves at the surface of a polytropic star

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# Motivation

The problem:

- 3D simulations of stars as fluid balls use an unphysical "atmosphere" to avoid dividing by $\rho \to 0$.
- As $\rho \to 0$, the perfect fluid approximation is no longer justified physically (need plasma physics)...
- ...but taking it seriously is better than using an atmosphere.

This talk:

- What is the behaviour of $v^i$, $\rho$, etc. at the surface? What are the *kinematic* boundary conditions?
- $c \sim \sqrt{r_* - r} \to 0$: does this mean every outgoing sound wave shocks?

Future work:

- Generic behaviour with shocks
- Develop HRSC numerical methods that do not use an atmosphere

# Physical approximations

- Newtonian motion and gravity
- Polytropic perfect fluid $P = K\rho^\Gamma$ (no heat, no shocks)
- Spherical symmetry
- Near the surface:
  - planar symmetry
  - constant gravitational field $g$

$$\begin{aligned} v_t + v v_x + \frac{P_x}{\rho} &= -g, \\ \rho_t + v\rho_x + \rho v_x &= 0. \end{aligned}$$

# Static solution

- Polytropic index $\Gamma \equiv 1 + \frac{1}{n}$
- Units such that $\Gamma K = 1$ and perhaps $g = 1$.
- Static solution with star in $-\infty < x < 0$:

$$v = 0, \quad \rho = (-gx)^n$$

- Sound speed $c$ defined by $c^2 = \frac{dP}{d\rho}$:

$$c = \sqrt{-gx}$$

# Riemann invariants

- Replacing $\rho$ with $c$:

$$\begin{aligned} v_t + v v_x + 2n c c_x &= -g, \\ c_t + v c_x + \frac{1}{2n} c v_x &= 0. \end{aligned}$$

- Riemann invariants, including gravity:

$$[\partial_t + (v \pm c)\partial_x]\,(v + gt \pm 2nc) = 0.$$

# Hodograph transformation 1

Interchange $(t,x)$ with $(v,c)$.

$$\begin{pmatrix} \partial_v \\ \partial_c \end{pmatrix} = \begin{pmatrix} t_v & x_v \\ t_c & x_c \end{pmatrix} \begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix},$$

and its inverse

$$\begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix} = \Delta^{-1} \begin{pmatrix} x_c & -x_v \\ -t_c & t_v \end{pmatrix} \begin{pmatrix} \partial_v \\ \partial_c \end{pmatrix},$$

In particular, we have

$$\begin{pmatrix} v_t & c_t \\ v_x & c_x \end{pmatrix} = \Delta^{-1} \begin{pmatrix} x_c & -x_v \\ -t_c & t_v \end{pmatrix}.$$

# Hodograph transformation 2

- The system without gravity,

$$\begin{aligned} v_t + v v_x + 2n c c_x &= 0, \\ c_t + v c_x + \frac{1}{2n} c v_x &= 0, \end{aligned}$$

  is
  - first order, homogeneous, quasilinear, autonomous;
  - 2 equations for 2 unknowns.
- Under the hodograph transformation, it becomes linear:

$$\begin{aligned} x_c - v t_c + 2n c t_v &= 0, \\ -x_v + v t_v - \frac{1}{2n} c t_c &= 0, \end{aligned}$$

# Bringing the nonlinear problem into linear form

Case $n = 1 \Leftrightarrow$ shallow water equations on a sloping beach: Carrier & Greenspan 1958

- Independent variables $c$ and $\lambda = v + gt$

$$v_{\lambda\lambda} = v_{cc} + \frac{2n+1}{c} v_c .$$

- Free boundary is now fixed at $c = 0$.
- The nonlinearity is in the transformation

$$\begin{aligned} v_x &= \Delta^{-1} v_c , \\ c_x &= \Delta^{-1} (1 - v_\lambda), \\ \text{where } \Delta &= -\sigma \left[ (1 - v_\lambda)^2 - v_c^2 \right] \end{aligned}$$

- Shocks only as $\Delta \to 0 \Rightarrow$ No shocks for small enough $v$.

# New result 1: Eulerian boundary conditions

Regular solutions, linear superpositions of

$$v(\lambda, c) = e^{i\omega\lambda} c^{-n} J_n(\omega c)$$

- are regular and even in $c$
- obey the boundary condition $v_c = 0$ at $c = 0$.

Near the boundary $\Delta \sim c$, hence

- $c_x \sim c^{-1}(1 - v_\lambda) \quad \Rightarrow \quad c^2(x, t) = \text{(regular in } x\text{)}$
- $v_x \sim c^{-1} v_c \quad \Rightarrow \quad v(x, t) = \text{(regular in } x\text{)}$

(but they are *not* even or odd in $x_* - x$).

# New result 2: Criterium for no shocks

- Consider wave packet at position $x_0$, width $x_1$, amplitude $v_0$, still in the linear regime.
- Transform into initial data for $v(\lambda, c)$.
- To leading order the outgoing wave is

$$v(\lambda, c) \simeq c^{-n-\frac{1}{2}} f(c+\lambda)$$

- Wave packet turns around at $c \sim c_1$.
- For $\Delta > 0$ we need

$$|v_c| \sim |v_\lambda| \sim \frac{v_0}{c_1}\left(\frac{c_1}{c_0}\right)^{-n-\frac{1}{2}} \lesssim 1.$$

- Equivalent to

$$\frac{v_0}{v_*} \lesssim \left(\frac{x_1}{x_0}\right)^{n+1} \left(\frac{x_0}{r_*}\right)^{\frac{1}{2}}$$

where $v_* = \sqrt{2gr_*}$ is the escape velocity.