How to slice a black hole safely

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Outline

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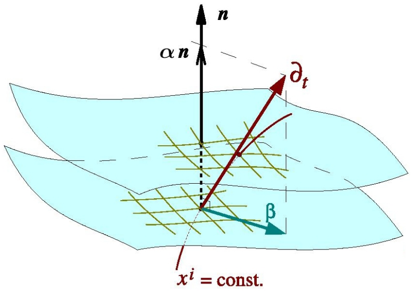
3+1 decomposition

- the coordinates and the shape of the time slices are determined by the **lapse** α and the **shift** β^i
- these gauge quantities are defined by additional *gauge conditions*



(from E. Gourgoulhon, Lecture notes)

- consider Bona-Massó slicing conditions (Bona *et. al* 1995)

$$(\partial_t - \mathcal{L}_\beta)\alpha = -\alpha^2 f(\alpha) K$$

Slicing conditions

$\longrightarrow$ uniquely define the geometry of the three-dimensional slices

popular slicing conditions

name	equation	$f(\alpha)$
geodesic slicing	$\alpha = 1$	$f = 0$
maximal slicing	$K = 0$	"$f \to \infty$"
harmonic slicing	$(\partial_t - \mathcal{L}_\beta)\alpha = -\alpha^2 K$	$f = 1$
1+log slicing	$(\partial_t - \mathcal{L}_\beta)\alpha = -2\alpha K$	$f = 2/\alpha$

In order to understand some of their properties, we study these conditions in the case of a *single Schwarzschild black hole*.

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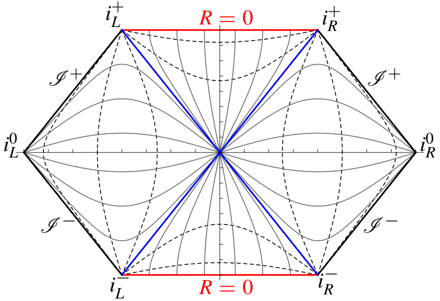
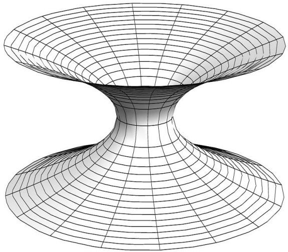
Schwarzschild black hole





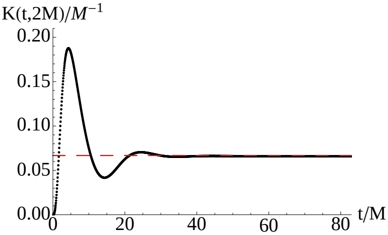
- (T, R, θ, φ) standard Schwarzschild coordinates
- initial slice at $T =$ const. in spatially isotropic coordinate

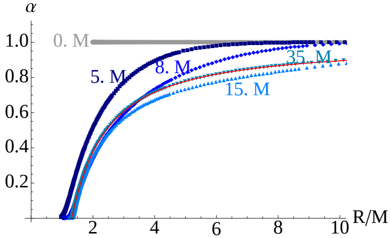
$$^{(3)}ds^2 = \psi^4(dr^2 + r^2\, d\Omega^2)$$

with $\psi = 1 + \dfrac{M}{2r}, \quad R = \psi^2\, r$

Numerical Observations

Numerical simulation of a Schwarzschild black hole in full 3D, using "1+log" slicing condition and "Gamma-driver" shift condition:





$\longrightarrow$ Invariant quantities settle down, approaching asymptotically a *stationary state*?

But: There is no time-independent wormhole/puncture slicing of Schwarzschild with an everywhere positive lapse!
(Hannam, Evans, Cook, Baumgarte, 2003 / Reimann, Brügmann 2003)

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The height function approach

The Schwarzschild spacetime

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dT^2 + \left(1 - \frac{2M}{R}\right)^{-1} dR^2 + R^2 d\Omega^2$$

Consider time-independent foliations with spatial coordinates (R, θ, φ):

The height function h

$$T = t + h(R)$$

- $\Rightarrow$ use this ansatz to calculate $g_{\mu\nu} = g_{\mu\nu}(R, h')$
- $\Rightarrow$ express all relevant quantities $(K, \alpha, \beta^R, \ldots)$ in a convenient way

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Bona-Massó type slicing

- all stationary states of Schwarzschild are characterised by $\gamma_{RR} = 1/\alpha^2$ and

Killing lapse and shift

$$\alpha^2 - \beta^i \beta_i = 1 - \frac{2M}{R}$$

Extrinsic curvature

$$K = \beta' + \frac{2\beta}{R} \qquad (\beta = \sqrt{\beta^R \beta_R})$$

Stationary states for Bona-Massó conditions

$$\beta^R \alpha' = \alpha^2 f(\alpha) K$$

$$\Rightarrow \boxed{\alpha^2 = 1 - \frac{2M}{R} + R^{-4} \exp\left(2 \int \frac{d\alpha}{\alpha f(\alpha)}\right)}$$

$$\Leftrightarrow \quad (\alpha^2 - 1) R^4 + 2MR^3 - \exp\left(2 \int \frac{d\alpha}{\alpha f(\alpha)}\right) = 0$$

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Stationary States

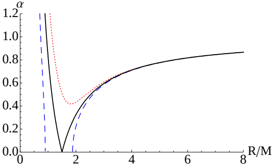
Maximal slicing

(Estabrook *et al.* 1973)

$$K = \beta' + \frac{2\beta}{R} = 0$$

$$\alpha = \sqrt{1 - \frac{2M}{R} + \frac{C^2}{R^4}}$$

$$C^2_{\text{crit}} = \frac{27M^4}{16}$$



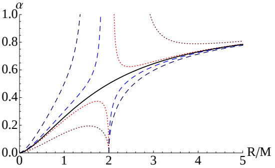
Harmonic slicing

(Cook & Scheel 1997)

$$\beta^R \alpha' = \alpha^2 K$$

$$\alpha = \sqrt{\frac{1 - \frac{2M}{R}}{1 - \frac{C^2}{R^4}}}$$

$$C^2_{\text{crit}} = (2M)^4$$



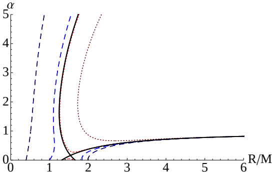
1+log slicing

(Hannam *et al.* 2006)

$$\beta^R \alpha' = 2\alpha K$$

$$\alpha^2 = 1 - \frac{2M}{R} + \frac{C^2 e^{\alpha}}{R^4}$$

$$C^2_{\text{crit}} = \frac{(3+\sqrt{10})^3 M^4}{128 e^{\sqrt{10}-3}}$$

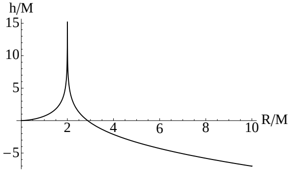


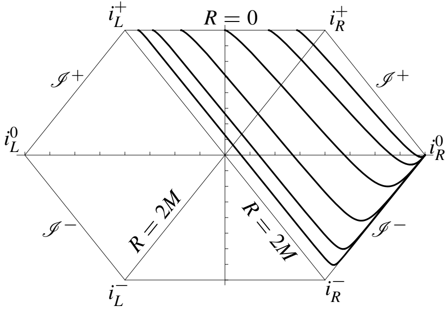
Carter-Penrose diagrams

Geodesic slicing

The height function

$$h' = -\frac{\beta/\alpha}{1-\frac{2M}{R}}$$





- Singularity $R = 0$ at finite proper distance

$\longrightarrow$ Not singularity avoiding

Carter-Penrose diagrams

Harmonic slicing

The height function

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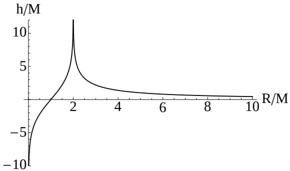

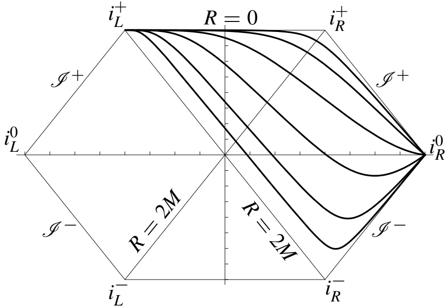

- Singularity $R = 0$ at infinite proper distance
- Arbitrarily small R covered

$\longrightarrow$ Marginally singularity avoiding

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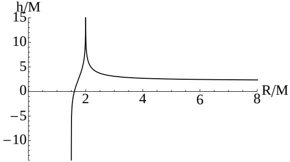
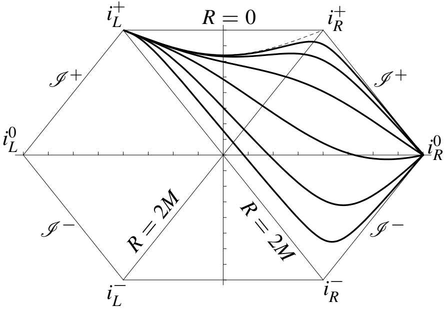
Carter-Penrose diagrams

Maximal slicing

The height function

$$h' = -\frac{\beta/\alpha}{1 - \frac{2M}{R}}$$





- Root $R_0 = 3M/2$ at infinite proper distance
- Only $R > R_0$ covered

$\longrightarrow$ Strongly singularity avoiding

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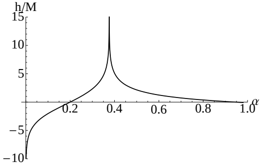
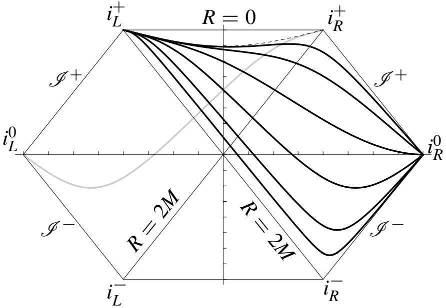
Carter-Penrose diagrams

1+log slicing

The height function

$$h' = -\frac{\beta/\alpha}{1 - \frac{2M}{R}}$$



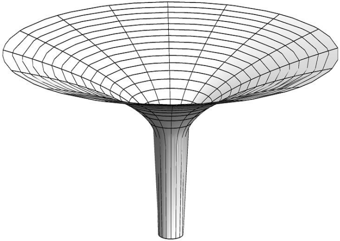


- Root $R_0 \approx 1.312M$ at infinite proper distance
- Only $R > R_0$ covered

$\longrightarrow$ Strongly singularity avoiding

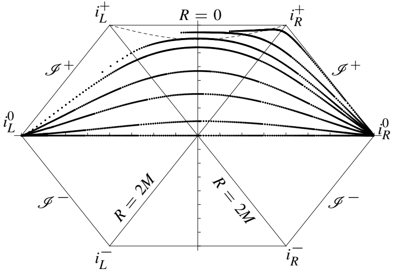
"The trumpet"

- the geometry of the stationary states in the case of strongly singularity avoiding conditions (maximal, 1+log) is described by a *trumpet*



Question: How is the transition from a wormhole to a trumpet possible?

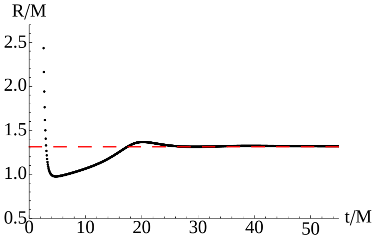
Answer: only on the numerical grid!



(displayed times:

$t/M = 0.25, 0.75, 1.25, 2., 2.5, 3.5, 8)$

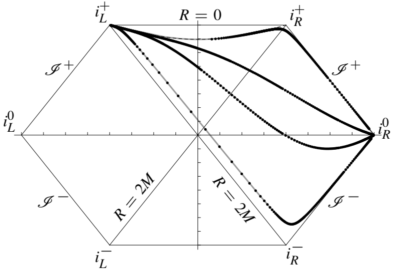
R near the puncture



(grid point at $r = 6M/1024$)

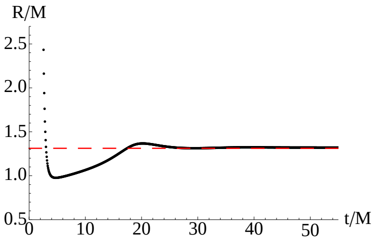
- points are dragged to the right-hand side of the diagram by the Gamma-driver shift $\partial_t^2 \beta^i = 3/4\, \partial_t \tilde{\Gamma}^i - \eta\, \partial_t \beta^i$
- the slices stretch along $R = R_0$ due to the slicing condition (1+log)

Answer: only on the numerical grid!



(displayed times: $t/M = 30, 37.25, 40, 50$
initial slice at $T_0 = -40M$)

R near the puncture



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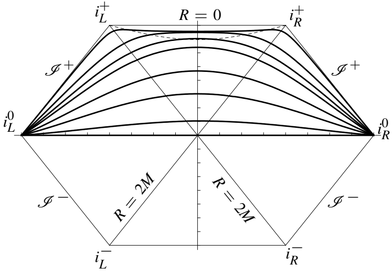
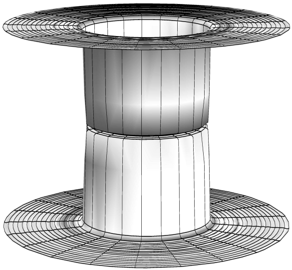


(obtained by $\beta^i = 0$)



- slices always extend from i_L^0 to i_R^0
- certain range of $R < R_0$ is covered
- two non-trivially connected trumpets are formed asymptotically

Stationary state in isotropic coordinates

Transformation

$$ {}^{(3)}ds^2 = \alpha^{-2}\,\gamma_{RR} + R^2\,d\Omega^2 \quad\longleftrightarrow\quad {}^{(3)}ds^2 = \psi^4(r)\,(dr^2 + r^2\,d\Omega^2) $$

$\longrightarrow$ with the given function $R(\alpha)$ we integrate either

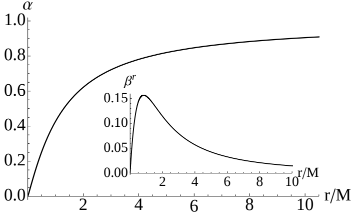
$$ r(\alpha) = R(\alpha)^{1/\alpha} \exp\left(- \int_\alpha^1 \frac{\ln R(\bar\alpha)}{\bar\alpha^2}\, d\bar\alpha \right) $$

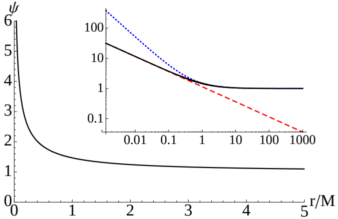
or

$$ r(\alpha) = \exp\left(\int^\alpha \frac{1}{\bar\alpha R(\bar\alpha)} \frac{dR}{d\alpha}(\bar\alpha)\, d\bar\alpha \right) $$

to obtain $\psi = \sqrt{\frac{R}{r}},\ \beta^r = \frac{\partial r}{\partial R}\beta^R, \ldots$

Stationary state



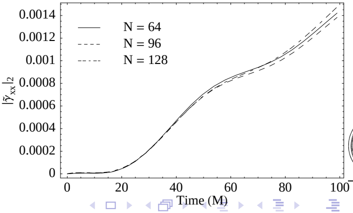


$\psi \sim \sqrt{R_0/r}$ for $r \to 0$

$\psi \sim 1 + M/(2r)$ for $r \to \infty$

- used as initial data
- numerically evolved with 1+log and $\tilde{\Gamma}$-driver shift

$\Rightarrow$ explicitly time-independent data



Conclusion

- There are non-trivial stationary states of the Schwarzschild spacetime with an overall positive lapse that satisfy certain slicing conditions.
- The stationary states are numerically relevant, as appropriate shift conditions allow the *numerical slices* to asymptote to these states.
- Trumpet initial data minimise the gauge evolutions and provide an excellent test bed for moving puncture codes.

Future work

- mathematical issues (stability, relevance of solutions with other integration constants, ...)
- go beyond spherical symmetry
    - e.g. stationary states in axisymmetry
    - generalised trumpet initial data (binaries?)
- improved gauge conditions?