Outer Boundary Conditions for Einstein's Equation in Harmonic Gauge

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Outline

1. **Motivation**
   - Solve Einstein's field equation on a bounded domain
2. **Harmonic Gauge**
   - Einstein Field's Equation
3. **Boundary conditions**
   - Well posedness
4. **The Quality of the boundary conditions**
5. **Conclusions**

Outline

Why do we need other boundary conditions?

Problem

Many evolutions of dynamical phenomena are not efficient.

Example

- Typically: Numerical evolution of binary system uses outer boundary conditions at $\sim$ 300-400M, and the wave extraction sphere at $\sim$ 40-60M.

Expensive!

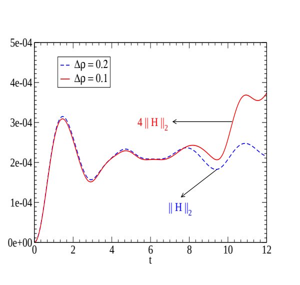
Why do we need other boundary conditions?

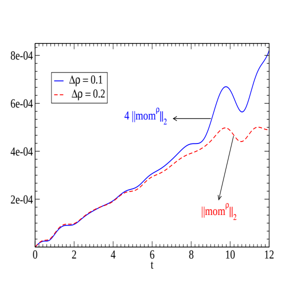
Problem

Many evolutions of dynamical phenomena are not efficient.

Example

- Axisymmetric evolution of Minkowsky with radiative boundary conditions





Why do we need other boundary conditions?

Problem

Many evolutions of dynamical phenomena are not efficient.

Solutions

Actually we have different ways

- adaptive mesh refinement,
- coordinate transformation will not affect physics (fisheye),
- good boundary conditions

Good boundary condition

Good boundary conditions should

- preserve the constraints,
- yield a well-posed initial value problem
- minimize spurious reflection at the boundary.

What kind of boundaries?

Absorbing outer boundary conditions!

Good boundary condition

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An example

- Scalar wave equation

$$(\partial_t^2 - \partial_x^2)u = 0\,, \qquad x \in [-a, a]\,.$$

Then, the general solution is

$$u(t,x) = f(x+t) + f(t-x)\,.$$

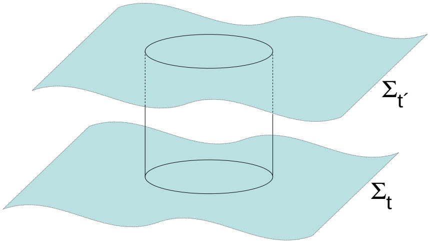
Therefore, the boundary conditions

$$(\partial_t - \partial_x)u(t,-a) = 0\,, \quad (\partial_t + \partial_x)u(t,a) = 0\,,$$

are perfectly absorbing.

Outer boundary in general relativity

In numerical relativity, one typically considers a compact spatial domain with boundary



Radiative boundary conditions

$$\partial_t F + v\,\partial_r F + v\,(F - F_0) \sim 0\,.$$

Outline

Einstein Field's Equation in the Harmonic Gauge

- Harmonic gauge condition

$$\Box x^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = 0\,. \quad (\text{restrictive!})$$

- Generalized harmonic gauge condition

$$\Box x^\mu = H^\mu \quad \Rightarrow \quad \mathcal{C}^\mu \equiv g^{\mu\nu}\Gamma^\alpha_{\mu\nu} + H^\alpha = 0\,,$$

  where H^α are given functions.
- Evolution equations

$$\Box g_{\mu\nu} \simeq 0\,.$$

- Another generalization

$$\mathcal{C}^{\mu} \equiv g^{\mu\nu}(\Gamma^{\alpha}_{\mu\nu} - \hat{\Gamma}^{\alpha}_{\mu\nu}) + H^{\alpha} = 0\,,$$

where $\hat{g}_{\mu\nu}$ is a fixed background.

- Evolution equations

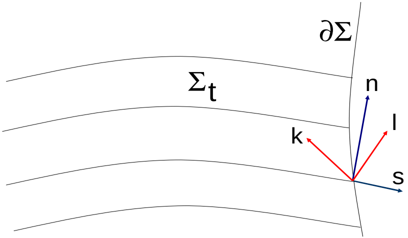
$$\hat{g}^{\mu\nu}\hat{\nabla}_{\mu}\hat{\nabla}_{\nu}g_{\alpha\beta} = S_{\mu\nu}\,,$$

where $S_{\mu\nu}$ depends on first derivatives of $g_{\mu\nu}$.

Ten wave equation $\Rightarrow$ ten boundary conditions

Outline

To specify boundary conditions, we need to construct a tetrad at the boundary:



- n^μ: normal vector to the slices
- s^μ: normal vector to the boundary
- Null vectors

$$k^\mu = \frac{1}{\sqrt{2}}(n^\mu - s^\mu)\,, \quad l^\mu = \frac{1}{\sqrt{2}}(n^\mu + s^\mu)\,,$$

First order boundary conditions

Constraint preserving boundary conditions

$$l^\mu \mathcal{C}_\mu \doteq 0\,,$$

4 conditions

Boundaries in terms of the shear of the outgoing null congruence

$$m^\mu\, m^\nu \sigma^{(l)}_{\mu\nu} \doteq q_2 \quad \Rightarrow \quad \hat{\nabla}_l g_{mm} - 2\,\hat{\nabla}_m g_{lm} \doteq 2\left(q_2 - K^{(l)}_{mm}\right)\,.$$

2 conditions

- Other 4 conditions, with some data: $\hat{\nabla}_l g_{ll} \doteq p$, $\hat{\nabla}_l g_{lk} \doteq \pi$, $\hat{\nabla}_l g_{lm} \doteq q_1$.

Second order boundary conditions

Constraint preserving boundary conditions

$$l^\mu \hat{\nabla}_\mu \mathcal{C}_\nu \doteq 0\,,$$

4 conditions

Radiation controlling boundary conditions

$$\Psi_0 = C_{\mu\nu\alpha\beta} l^\mu m^\nu l^\alpha m^\beta \doteq q_2'.$$

2 conditions

- Other 4 conditions, with some data: $\hat{\nabla}_l \hat{\nabla}_l g_{ll} \doteq p'$, $\hat{\nabla}_l \hat{\nabla}_l g_{lk} \doteq \pi'$, $\hat{\nabla}_l \hat{\nabla}_l g_{lm} \doteq q_1'$.

Initial value problem

One can show the well posedness using the frozen coefficient approximation.

- System reduces to a linear problem with constant coefficient on half space $x > 0$.
- With a coordinate transformation, one can rewrite the metric as

$$\hat{g}_{\mu\nu} = -dt^2 + (dx + \beta\, dt)^2 + dy^2 + dz^2 \,,$$

- We obtain a system of ten wave equation

$$\begin{aligned} \left(\partial_t^2 \right. &+ \left. 2\beta\,\partial_t\partial_x + (1+\beta)\partial_x^2 + \partial_y^2 + \partial_z^2\right) g_{\mu\nu} = \mathcal{F}_{\mu\nu} \,, \\ \left(\partial_t \right. &- \left. (1+\beta)\partial_x\right)^2 \doteq \amalg_{\mu\nu} \,, \end{aligned}$$

$\amalg_{\mu\nu}$ depends on tangential derivative of $g_{\mu\nu}$

Difficulty

The sources of wave equations depends on the first order tangential derivatives of $g_{\mu\nu}$. One cannot use the standard energy estimation.

- Using the Kreiss theory, one can get an estimation for the solution and can show that the problem is well posed in the frozen approximation.

Outline

Numerical test

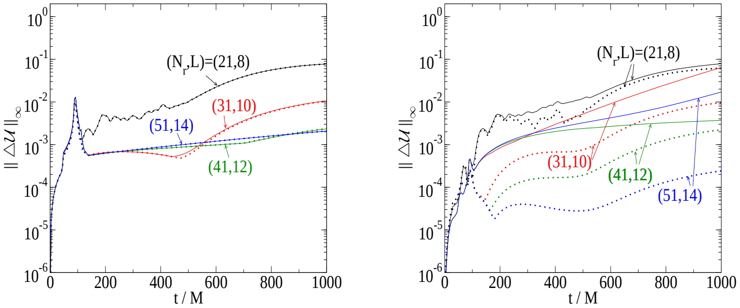
- Schwarzschild black hole + outgoing quadrupolar gravitational wave,
- H_μ are chosen initial such that the time derivative of α and β vanish.
- Boundary conditions
  1. the first outer boundary conditions:
  2. the Kreiss-Winicour boundary conditions (gr-qc/0602051)

$$\hat{\nabla}_l g_{mm} - 2\,\hat{\nabla}_m g_{lm} \doteq 2\left(q_2 - K^{(l)}_{mm}\right) \;\Rightarrow\; \hat{\nabla}_l g_{mm} \doteq q_2\,.$$

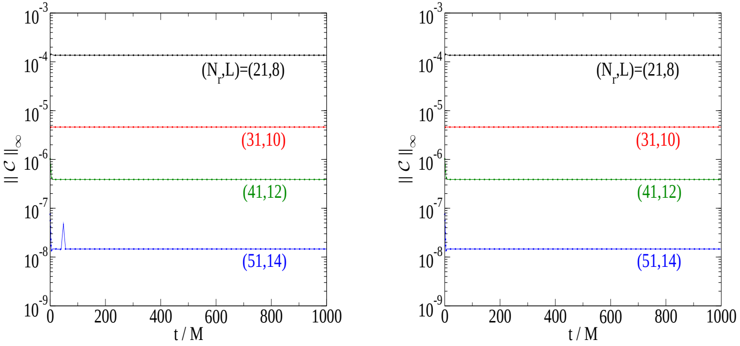
  3. second order boundary conditions
  4. mixes outer boundary conditions: the previous one but with $\hat{\nabla}_l g_{ll} \doteq p$, $\hat{\nabla}_l g_{lk} \doteq \pi$, $\hat{\nabla}_l g_{lm} \doteq q_1$.
- Evolve reference solution u^{ref}: $1.9 \leq r \leq 961.9M$
- Evolve solution u: $1.9 \leq r \leq 41.9M$

- Code SpEC, which is based on a pseudospectral collocation method.
- $\Phi_{i\mu\nu} \equiv \partial_i g_{\mu\nu}$, $\Pi \equiv -t^\alpha \partial_\alpha g_{\mu\nu}$
- N_r Chebyshev coefficients, $l \leq L$ for spherical harmonic
- Monitor the difference:

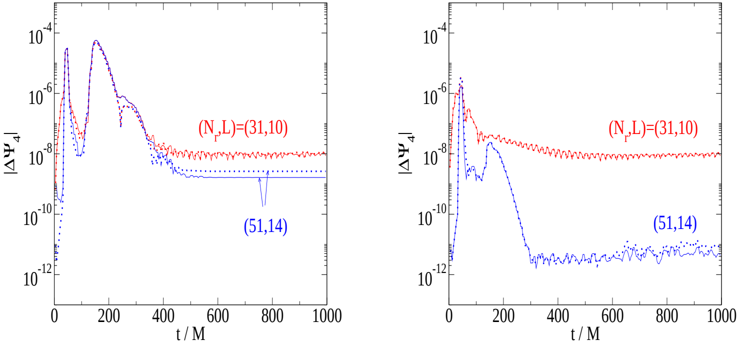
$$\Delta\mathcal{U} \equiv \delta^{\mu\nu}\,\delta^{\alpha\beta}\left[\Delta g_{\mu\nu}\,\Delta g_{\alpha\beta} + \Delta\Pi_{\mu\nu}\,\Delta\Pi_{\alpha\beta} + \Delta\Phi_{\mu\nu}\,\Delta\Phi_{\alpha\beta}\right]^{1/2},$$



- Monitor the constraints



- What about gravitational waves?



Outline

Conclusions

- We have presented two sets of absorbing and constraint preserving boundary conditions (first and second order conditions)
- The second order boundary conditions enable one to fix the scalar Ψ_0 at the boundary. We have, in some sense, the control of the incoming gravitational radiation.
- One can show that those conditions form a well-posed initial boundary value problem.