Non-existence of stationary two-black-hole configurations

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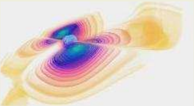

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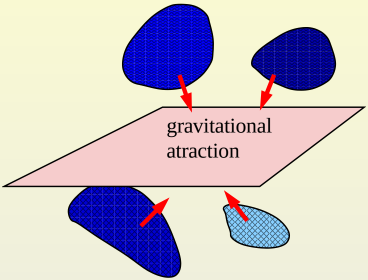
Introduction

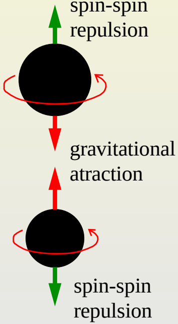
Newtonian Theory



- *Classical result:* No static n-body configuration (with bodies separated by a plane and $n > 1$)

General Relativity

- No static reflectionally symmetric n-body configuration with $n > 1$ [Beig and Schoen, 2009]
- Does the spin-spin interaction change the picture?
- No symmetric 2-body configuration with anti-aligned spins [Beig, Gibbons, Schoen, 2009]
- *Here:* Two aligned axisymmetric black holes in equilibrium?



Introduction

- *Extensively discussed candidate for two aligned black holes:*
  The double-Kerr-NUT solution [Kramer and Neugebauer, 1980]
- *Questions:*
  - Is the double-Kerr-NUT solution the only candidate?
    $\rightarrow$ Yes!
    (Rigorous solution of a boundary value problem)
  - Is this solution “physically reasonable”?
    $\rightarrow$ No!
    (Violates $8\pi|J| < A$ [J: angular momentum, A: horizon area] or $M > 0$ [M: ADM mass] and has singular rings outside the horizons)

Plan of the talk

1. The boundary value problem
2. Solution of the boundary value problem
3. Sub-extremal black holes
4. Degenerate black holes

Coordinates

- *Axisymmetric and stationary vacuum spacetime:*
  introduce Weyl coordinates $(\varrho, \zeta, \varphi, t)$ with the line element

$$\mathrm{d}s^2 = \mathrm{e}^{-2U}\left[\mathrm{e}^{2k}(\mathrm{d}\varrho^2 + \mathrm{d}\zeta^2) + \varrho^2\mathrm{d}\varphi^2\right] - \mathrm{e}^{2U}(\mathrm{d}t + a\,\mathrm{d}\varphi)^2,$$

  where $U = U(\varrho,\zeta),\ k = k(\varrho,\zeta),\ a = a(\varrho,\zeta)$

- *Killing vectors:*

$$\xi^i = \delta^i_t, \qquad \xi^i\xi_i < 0 \quad \text{(stationarity)}$$
$$\eta^i = \delta^i_\varphi, \qquad \eta^i\eta_i > 0 \quad \text{(axisymmetry)}$$

Boundary conditions

- *Event horizons:*
  - Axisymmetry, stationarity: Event horizon of a black hole is a Killing horizon

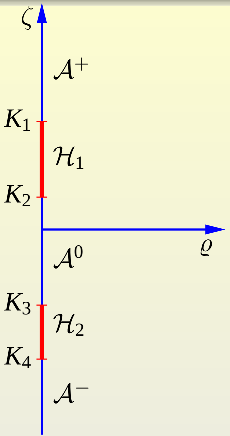
$$\mathcal{H}_1: \quad |\xi + \Omega_1 \eta| = 0$$
$$\mathcal{H}_2: \quad |\xi + \Omega_2 \eta| = 0$$

  Ω_1, Ω_2: Angular velocities of the horizons
  - In Weyl Coordinates: Event horizon degenerates to a "straight line" [Carter, 1973]:

$$\mathcal{H}_1: \quad \varrho = 0, \quad K_1 \geq \zeta \geq K_2$$
$$\mathcal{H}_2: \quad \varrho = 0, \quad K_3 \geq \zeta \geq K_4$$

- *Infinity:* Spacetime is asymptotically flat

$$\varrho^2 + \zeta^2 \to \infty: \quad U \to 0, \quad k \to 0, \quad a \to 0$$



- *Axis of symmetry:* Regularity

$$\mathcal{A}^+, \mathcal{A}^0, \mathcal{A}^-: \quad a = 0, \quad k = 0$$

The Einstein equations — Ernst formulation

- *Introduce complex Ernst potential:*

$$f(\varrho,\zeta) = \mathrm{e}^{2U(\varrho,\zeta)} + \mathrm{i}b(\varrho,\zeta),$$

where the twist potential b is related to the metric potential a via

$$a_{,\varrho} = \varrho \mathrm{e}^{-4U} b_{,\zeta}, \qquad a_{,\zeta} = -\varrho \mathrm{e}^{-4U} b_{,\varrho}$$

- *Field equations:* Equivalent to Ernst equation [Ernst, 1968]

$$(\Re f)\left(f_{,\varrho\varrho} + f_{,\zeta\zeta} + \tfrac{1}{\varrho} f_{,\varrho}\right) = f_{,\varrho}^2 + f_{,\zeta}^2$$

- The remaining metric potential k can be calculated from f by a line integral via

$$\begin{aligned} k_{,\varrho} &= \varrho\left[U_{,\varrho}^2 - U_{,\zeta}^2 + \frac{1}{4}\mathrm{e}^{-4U}(b_{,\varrho}^2 - b_{,\zeta}^2)\right], \\ k_{,\zeta} &= 2\varrho\left[U_{,\varrho}U_{,\zeta} + \frac{1}{4}\mathrm{e}^{-4U} b_{,\varrho} b_{,\zeta}\right]. \end{aligned}$$

The linear problem

- *Complex coordinates:*

$$z=\varrho+\mathrm{i}\zeta,\quad \bar z=\varrho-\mathrm{i}\zeta$$

- *The linear problem for the axisymmetric and stationary Einstein vacuum equations:*
  system of first order equations for a 2×2 matrix pseudopotential $\boldsymbol{\Phi}=\boldsymbol{\Phi}(K,z,\bar z)$ [Neugebauer, 1979, 1980]

$$\boldsymbol{\Phi}_{,z}=\left[\begin{pmatrix} B & 0\\ 0 & A\end{pmatrix}+\lambda\begin{pmatrix} 0 & B\\ A & 0\end{pmatrix}\right]\boldsymbol{\Phi}$$

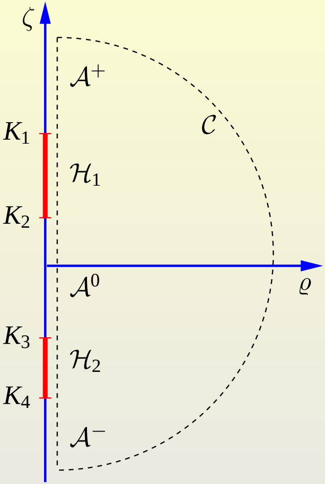
$$\boldsymbol{\Phi}_{,\bar z}=\left[\begin{pmatrix} \bar A & 0\\ 0 & \bar B\end{pmatrix}+\frac{1}{\lambda}\begin{pmatrix} 0 & \bar A\\ \bar B & 0\end{pmatrix}\right]\boldsymbol{\Phi}$$

$$\lambda(K,z,\bar z)=\sqrt{\frac{K-\mathrm{i}\bar z}{K+\mathrm{i}z}},\quad A=\frac{f_{,z}}{f+\bar f},\quad B=\frac{\bar f_{,z}}{f+\bar f}$$

$K\in\mathbb{C}$: spectral parameter

- *Integrability condition:* $\boldsymbol{\Phi}_{,z\bar z}=\boldsymbol{\Phi}_{,\bar z z}$ $\Leftrightarrow$ Ernst equation

Axis and infinity I



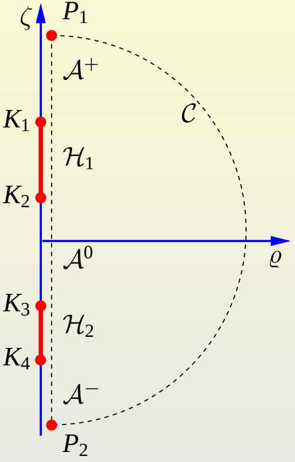
- *First step:* Integration of the LP along $\mathcal{A}^+, \mathcal{C}, \mathcal{A}^-, \mathcal{H}_2, \mathcal{A}^0, \mathcal{H}_1$
- *ζ-axis:* $\varrho = 0 \Rightarrow \lambda = \sqrt{\frac{K-\zeta-\mathrm{i}\varrho}{K-\zeta+\mathrm{i}\varrho}} = \pm 1$
  $\Rightarrow$ LP reduces to ODE with solution

  $$\boldsymbol{\Phi}(\varrho = 0, \zeta, \lambda = 1) = \begin{pmatrix} \bar{f}(\zeta) & 1 \\ f(\zeta) & -1 \end{pmatrix} \mathbf{C}(K)$$

  ($\mathbf{C}$: 2×2 matrix “integration constant”)
  5 different integration constants on $\mathcal{A}^-, \mathcal{H}_2, \mathcal{A}^0, \mathcal{H}_1, \mathcal{A}^+$
- *“Infinity” $\mathcal{C}$:* $\boldsymbol{\Phi} = \boldsymbol{\Phi}(K)$
  ($\boldsymbol{\Phi}$ is independent of ϱ, ζ)

Axis and infinity II

- Continuity of $\boldsymbol{\Phi}$ at K_1, K_2, K_3, K_4, P_1, P_2, together with continuity of $\boldsymbol{\Phi}_1$ at K_1, K_2 and continuity of $\boldsymbol{\Phi}_2$ at K_3, K_4 leads to an algebraic system of equations for the components of the **C**-matrices
- $\boldsymbol{\Phi}_i$, $i=1,2$: Solution of the LP in a frame co-rotating with the horizon $\mathcal{H}_i$ $(\varrho'=\varrho,\ \zeta'=\zeta,\ \varphi'=\varphi-\Omega_i t,\ t'=t)$
- $\boldsymbol{\Phi}_i$ can be calculated from $\boldsymbol{\Phi}$:



$$\boldsymbol{\Phi}_i=\left[\begin{pmatrix} c_- & 0\\ 0 & c_+\end{pmatrix} + \mathrm{i}(K+\mathrm{i}z)\Omega_i \mathrm{e}^{-2U}\begin{pmatrix} -1 & -\lambda\\ \lambda & 1\end{pmatrix}\right]\boldsymbol{\Phi}$$

where $c_\pm := 1+\Omega_i(a\pm\varrho \mathrm{e}^{-2U})$

Axis potential

- From the solution of the algebraic system of equations it follows that the Ernst potential on $\mathcal{A}^+$ has the form

$$f^+(\zeta) = \frac{n_2(\zeta)}{d_2(\zeta)}, \quad n_2, d_2\text{: polynomials of second degree}$$

- *Determinant representation:* in terms of $\alpha_i := \frac{\bar{d}_2(K_i)}{d_2(K_i)}$ we obtain

$$f^+(\zeta) = \frac{\begin{vmatrix} 1 & K_1^2 & K_2^2 & K_3^2 & K_4^2 \\ 1 & \alpha_1 K_1(\zeta - K_1) & \alpha_2 K_2(\zeta - K_2) & \alpha_3 K_3(\zeta - K_3) & \alpha_4 K_4(\zeta - K_4) \\ 0 & K_1 & K_2 & K_3 & K_4 \\ 0 & \alpha_1(\zeta - K_1) & \alpha_2(\zeta - K_2) & \alpha_3(\zeta - K_3) & \alpha_4(\zeta - K_4) \\ 0 & 1 & 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & K_1^2 & K_2^2 & K_3^2 & K_4^2 \\ -1 & \alpha_1 K_1(\zeta - K_1) & \alpha_2 K_2(\zeta - K_2) & \alpha_3 K_3(\zeta - K_3) & \alpha_4 K_4(\zeta - K_4) \\ 0 & K_1 & K_2 & K_3 & K_4 \\ 0 & \alpha_1(\zeta - K_1) & \alpha_2(\zeta - K_2) & \alpha_3(\zeta - K_3) & \alpha_4(\zeta - K_4) \\ 0 & 1 & 1 & 1 & 1 \end{vmatrix}}$$

→ Axis potential of a (particular) double-Kerr-NUT solution

→ The double-Kerr-NUT solution is the only candidate for a black hole equilibrium configuration!

The double-Kerr-NUT solution

- *Ernst potential for all ϱ, ζ:*

$$f(\varrho,\zeta)=\frac{\begin{vmatrix} 1 & K_1^2 & K_2^2 & K_3^2 & K_4^2 \\ 1 & \alpha_1 K_1 r_1 & \alpha_2 K_2 r_2 & \alpha_3 K_3 r_3 & \alpha_4 K_4 r_4 \\ 0 & K_1 & K_2 & K_3 & K_4 \\ 0 & \alpha_1 r_1 & \alpha_2 r_2 & \alpha_3 r_3 & \alpha_4 r_4 \\ 0 & 1 & 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & K_1^2 & K_2^2 & K_3^2 & K_4^2 \\ -1 & \alpha_1 K_1 r_1 & \alpha_2 K_2 r_2 & \alpha_3 K_3 r_3 & \alpha_4 K_4 r_4 \\ 0 & K_1 & K_2 & K_3 & K_4 \\ 0 & \alpha_1 r_1 & \alpha_2 r_2 & \alpha_3 r_3 & \alpha_4 r_4 \\ 0 & 1 & 1 & 1 & 1 \end{vmatrix}}$$

$$r_i := \sqrt{\varrho^2+(\zeta-K_i)^2}, \quad K_i\in\mathbb{R}, \quad \alpha_i\in\mathbb{C}, \quad \alpha_i\bar{\alpha}_i=1,$$
$$i=1,\dots,4$$

Equilibrium conditions

- *Regularity on the symmetry axis:*
  - $k = 0$:

$$\alpha_1\alpha_2 + \alpha_3\alpha_4 = 0$$

  - $a = 0$:

$$\frac{(1-\alpha_4)^2}{\alpha_4}w^2 = \frac{(1-\alpha_3)^2}{\alpha_3}, \qquad w := \sqrt{\frac{(K_1-K_4)(K_2-K_4)}{(K_1-K_3)(K_2-K_3)}},$$

$$\frac{(1+\alpha_2)^2}{\alpha_2}w'^2 = \frac{(1+\alpha_1)^2}{\alpha_1}, \qquad w' := \sqrt{\frac{(K_2-K_3)(K_2-K_4)}{(K_1-K_3)(K_1-K_4)}},$$

- *Solution of the equilibrium conditions:*

$$\alpha_1 = \frac{w'\alpha^2 + \mathrm{i}\varepsilon\alpha}{w' - \mathrm{i}\varepsilon\alpha}, \quad \alpha_2 = \frac{\alpha^2 + \mathrm{i}w'\varepsilon\alpha}{1 - \mathrm{i}w'\varepsilon\alpha}$$

$$\alpha_3 = \frac{w\alpha^2 - \alpha}{w - \alpha}, \quad \alpha_4 = \frac{\alpha^2 - w\alpha}{1 - w\alpha}$$

$$\alpha \in \mathbb{C}, \quad \alpha\bar{\alpha} = 1, \quad w \in (0,1], \quad w' \in [1,\infty), \quad \varepsilon = \pm 1$$

$\rightarrow$ Free parameters: w, w', α (and, e.g., K_1, $K_2 - K_3$)

Properties of the two-horizon solution

- *Symmetric case impossible:*
  Two identical black holes → total (ADM) mass is negative
  [Hoenselaers and Dietz, 1983; Dietz and Hoenselaers, 1985]
- *At least one of the two Komar masses is negative:*
  - Conjecture based on numerical studies [Hoenselaers, 1984]
  - Rigorous proof [Manko, Ruiz and Sanabria-Gómez, 2000; Krenzer, 2000; Manko and Ruiz, 2001]

→ But: no criterion for exclusion

- Examples for regular, asymptotically flat configurations with negative Komar masses:
  - Central black hole with surrounding perfect fluid ring, $M_{\text{black hole}} < 0$
  - Central disk of dust with surrounding perfect fluid ring, $M_{\text{disk}} < 0$

  [Ansorg and Petroff, 2006]

Sub-extremal black holes

- *Kerr black hole:*
  - Solution depends on two parameters, e.g. M (ADM mass) and a (angular momentum per unit mass)
  - Black hole only for $|a| \leq M$
  - Sub-extremal Kerr black hole: $|a| < M$
  - Equivalent characterization: There are trapped surfaces inside the event horizon
- *Trapped surface:* Closed two-surface $\mathcal{S}$ on which outgoing light rays are converging $\Rightarrow$ negative expansion $\theta = h^{ab}\nabla_a l_b < 0$ (h: interior metric of $\mathcal{S}$, l: outgoing null vector field)
- *General black holes*: Sub-extremal black holes are characterized through existence of trapped surfaces in every sufficiently small interior neighbourhood of the event horizon [Booth and Fairhurst, 2008]

A universal inequality

- Every axisymmetric and stationary sub-extremal black hole satisfies the inequality (J: angular momentum, A: horizon area):

$$8\pi|J| < A$$

$\longrightarrow$ Can be shown by solving a variational problem
[JH, Ansorg and Cederbaum, 2008]

- *Remark:* Generalisation to charged black holes (Q: charge):

$$(8\pi J)^2+(4\pi Q^2)^2 < A^2$$

[JH, Cederbaum and Ansorg, 2009]

- *Test of the inequality for the two-horizon solution:*
  $\longrightarrow$ For every choice of the free parameters, at least one of the two black holes violates $8\pi|J| < A$.
  $\longrightarrow$ Two sub-extremal black holes cannot be in equilibrium.

Degenerate black holes

- *Degenerate black holes:* Defined by vanishing surface gravity κ,

$$\kappa = 0$$

$$(\mathcal{H}: \quad |\chi| \equiv |\xi + \Omega\eta| = 0, \quad \chi^i_{;j}\chi^j = \kappa\chi^i, \quad \kappa = \text{constant})$$

- *Assumption:* We can introduce the Weyl coordinates $(\rho, \zeta, \varphi, t)$ as in the non-degenerate case.
- *Consequences:* [Ansorg and Pfister, 2008]
  - The rotation rate $a := \frac{J}{M_C}$ reaches its maximum

$$|a| = M_C$$

$(M_C$: Christodoulou mass, $M_C := \sqrt{M_{\text{irr}}^2 + \frac{J^2}{4M_{\text{irr}}^2}}$, $M_{\text{irr}} := \sqrt{\frac{A}{16\pi}})$

$\longrightarrow$ extremal black hole

  - The inequality $8\pi|J| < A$ has to be replaced by

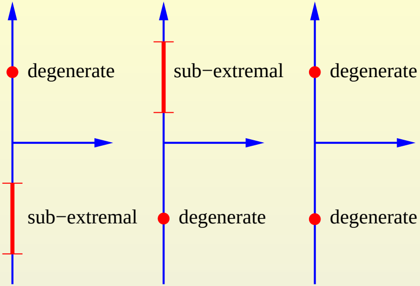
$$8\pi|J| = A$$

Degenerate black holes

- Characterization of degeneracy for the double-Kerr-NUT equilibrium solution:

  the upper black hole is degenerate
  $\Leftrightarrow w' = 1 \Leftrightarrow K_1 = K_2$

  the lower black hole is degenerate
  $\Leftrightarrow w = 1 \Leftrightarrow K_3 = K_4$



- Boundary value problem can be solved as before, but in different coordinates where the “horizon point” is blown up to a line again [Meinel et al. 2008]
- *Result:* The degenerate configurations are contained within the double-Kerr-NUT family of solutions, too.
- *Are these degenerate configurations physically reasonable?*
  - For most parameter choices: negative ADM mass, $M < 0$ ↯
  - In parameter region with positive ADM mass: singular ring in vacuum region ↯

Summary

- Two aligned black holes in equilibrium?
  Spin-spin repulsion compensates for gravitational attraction?
- Solution of boundary value problem via “Inverse method”:
  Particular double-Kerr-NUT solutions are the only candidates.
- Together with the inequality $8\pi|J| < A$ for sub-extremal black holes and the positive mass theorem $M > 0$ it follows that axisymmetric aligned two-black-hole configurations with
  - two sub-extremal black holes, or
  - one degenerate and one sub-extremal black hole, or
  - two degenerate black holes

  cannot be in equilibrium.
  [Neugebauer and JH, 2009, Gen. Relativ. Gravit. **41**, 2113]
  [JH and Neugebauer, in preparation]