Spectral analysis of quasi-normal modes of Schwarzschild black holes

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Contents

Hyperboloidal slices

- We consider hyperboloidal slices which penetrate the horizon $\mathcal{H}^+$ and include future null infinity $\mathscr{I}^+$.
- At $\mathscr{I}^+$ we may study outgoing radiation.
- Coordinate locations:
  $\mathscr{I}^+$: $\sigma = 0$; $\quad \mathcal{H}^+$: $\sigma = 1$
- Transformation to standard Schwarzschild coordinates:

$$r = \frac{2M}{\sigma},$$

$$t = 2M\left(2\tau + \frac{1}{\sigma} - \log[\sigma(1-\sigma)]\right)$$

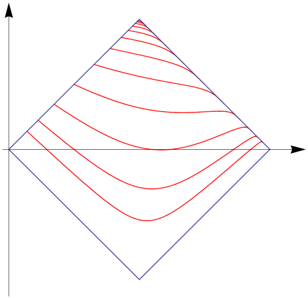


Figure: Penrose diagram

Teukolsky equation (TE)

- TE: linear wave equation for different kind of fields $\rightarrow$ *spin-weight* parameter λ
- solution's properties: quasi-normal ringing (QNM) $V(\tau,\sigma) \sim e^{-\gamma\tau}\sin(\omega\tau)$ and late-time tail $V(\tau,\sigma) \sim \tau^{-\mu}$
- Axial symmetry of Schwarzschild metric: separation of solution's angular part $\rightarrow$ *spin-weighted spherical harmonics* ${}_{\lambda}Y_{lm}(\vartheta,\varphi)$

$$
\begin{aligned}
0 = \ & \sigma^2(1-\sigma)V_{,\sigma\sigma} + [1-2\sigma^2]V_{,\tau\sigma} - (1+\sigma)V_{,\tau\tau} \\
& + \sigma[2-3\sigma+\lambda(2-\sigma)]V_{,\sigma} - [2\sigma+\lambda(\sigma-1)]V_{,\tau} \\
& - \sigma(1+\lambda)V + [\lambda(\lambda+1) - l(l+1)]V
\end{aligned}
$$

Laplace transformation

$$\tilde{V}(\sigma; s) := \mathcal{L}\{V(\tau,\sigma)\}(s) = \int_0^\infty e^{-s\tau} V(\tau,\sigma) d\tau,$$

$$\begin{aligned} &\sigma^2(1-\sigma)\tilde{V}_{,\sigma\sigma} + [s(1-2\sigma^2) + (2-3\sigma)\sigma + \lambda(2-\sigma)\sigma]\tilde{V}_{,\sigma} \\ &\quad -\{s^2(1+\sigma) + s[2\sigma + \lambda(\sigma-1)] + \sigma(\lambda+1) \\ &\qquad + l(l+1) - \lambda(\lambda+1)\}\tilde{V} \\ = &\quad (1-2\sigma^2)V_{0,\sigma} + (1+\sigma)[2\sigma + \lambda(\sigma-1)]V_0 \\ &\quad -(1-2\sigma^2)\dot{V}_0 - (1+\sigma)sV_0 \end{aligned}$$

Here: $\quad V_0 = V(\tau = 0, \sigma) \quad ; \quad \dot{V}_0 = (\partial_\tau V)(\tau = 0, \sigma)$

$$\rightarrow \quad \hat{A}(s)\tilde{V}(\sigma; s) = a + s \cdot b$$

- Asymptotical behaviour

$$\lim_{|s|\to\infty} \tilde{V}(\sigma; s) = 0,$$
$$\lim_{|s|\to\infty} s \cdot \tilde{V}(\sigma; s) = \lim_{\tau\to 0} V(\sigma, \tau)$$

- Inverse transformation:
  Bromwich integral

$$\begin{aligned} V(\sigma,\tau) &= \frac{1}{2\pi i}\int_\Gamma e^{s\tau}\tilde{V}(\sigma;s)ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\gamma+i\omega)\tau}\tilde{V}(\sigma;\gamma+i\omega)d\omega \end{aligned}$$

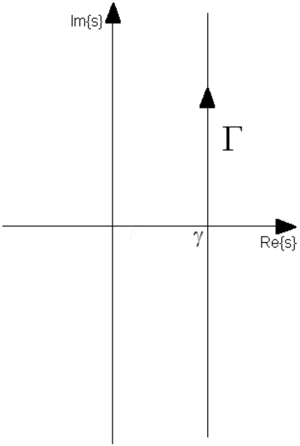


Figure: Bromwich integral curve

- Quantum mechanics: self-adjoint operators posess real eigenvalues with corresponding eigenvectors (discrete spectrum) as well as real improper eigenvalues with corresponding Dirac-vectors (continuous spectrum)

$$\hat{A}\,|\psi_k\rangle = \alpha_k\,|\psi_k\rangle\,, \quad \hat{A}\,|\psi(k)\rangle = \alpha(k)\,|\psi(k)\rangle$$

- Vectors can be written as:

$$|\phi\rangle = \sum_n c_n\,|\psi_n\rangle + \int c(n)\,|\psi(n)\rangle\,dn$$

- Is this also possible for solutions of the Teukolsky equation?

$$V(\tau,\sigma) \stackrel{?}{=} V_{QNM}(\tau,\sigma) + V_{tail}(\tau,\sigma)$$

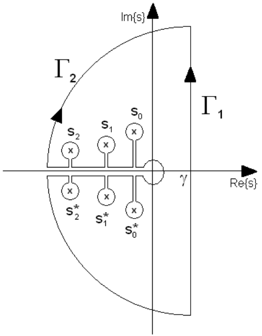


Figure: Deformed Bromwich integral

- Quartercircles ($s \to iz$)
  $\frac{1}{2\pi} \int_C \tilde{V}(\sigma; iz) e^{i\tau z} dz = 0$ if $\tau > 0$
  because of $\lim_{|s|\to\infty} \tilde{V}(\sigma; s) = 0$
  (Jordan's lemma)
- Negative real axis (branch cut)$\stackrel{?}{=}$ continuous spectrum with improper eigenvectors $\bar{V}(\sigma; s)$: $\eta(s) \leftrightarrow \{V_0, \dot{V}_0\}$?

$$V_{tail}(\tau, \sigma) := \frac{1}{2\pi i} \int_{-\infty}^{0} \tilde{V}(\sigma; s - i\epsilon) e^{\tau s} ds + \frac{1}{2\pi i} \int_{0}^{-\infty} \tilde{V}(\sigma; s + i\epsilon) e^{\tau s} ds$$

$$V_{tail}(\tau, \sigma) \stackrel{?}{=} \int_{-\infty}^{0} \eta(s) \bar{V}(\sigma; s) e^{s\tau} ds$$

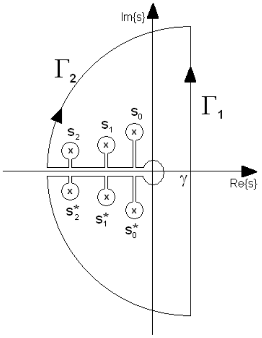


Figure: Deformed Bromwich integral

- Poles s_k: discrete spectrum with eigenvectors $\bar{V}_k(\sigma)$
- Around poles s_k: $\tilde{V}(\sigma; s) = \frac{\eta_k}{s-s_k}\bar{V}_k(\sigma) + \tilde{W}(\sigma) + \mathcal{O}(s-s_k)$
- Excitation coefficients η_k can be calculated directly from initial data $\{V_0, \dot{V}_0\} \to$ QNMs

$$\mathcal{L}^{-1}\left\{\frac{\eta_k}{s-s_k}\right\} = \eta_k e^{s_k\tau} \to$$

$$V_{QNM}(\tau,\sigma) = \sum_{k=0}^{\infty} 2Re\{\eta_k \bar{V}_k(\sigma) e^{s_k\tau}\}$$

QNM excitation coefficients

- Goal: derivation of QNM excitation coefficients η_k from initial data $\{V_0, \dot{V}_0\}$
- First: calculation of QNM frequencies s_k and eigenvectors $\bar{V}_k(\sigma)$:
  1. Solution of the homogeneous TE $\hat{A}(s_k)\bar{V}_k(\sigma) = 0$ with ansatz
  $$\bar{V}_k(\sigma) = \sum_{n=0}^{\infty} a_n(1-\sigma)^n$$
  2. Derivation of a three-term recurrence relation
  $$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0$$
  3. Continued fraction equation for $\{s_k\}$
- Works by Leaver and Cardoso

QNM excitation coefficients

- Inhomogeneous TE $\hat{A}(s)\tilde{V} = a + s \cdot b$ in pole proximity $(\{V_0, \dot{V}_0\} \to \{a, b\})$

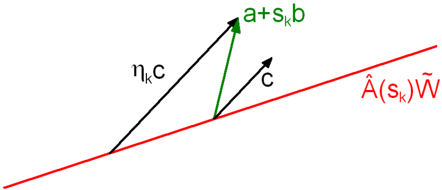
$$\tilde{V}(\sigma; s) = \frac{\eta_k}{s - s_k}\bar{V}_k(\sigma) + \tilde{W}(\sigma) + \mathcal{O}(s - s_k)$$

$$\hat{A}(s) = \hat{A}(s_k) + \left(\frac{\partial \hat{A}}{\partial s}\right)(s_k) \cdot (s - s_k) + \mathcal{O}(s - s_k)$$

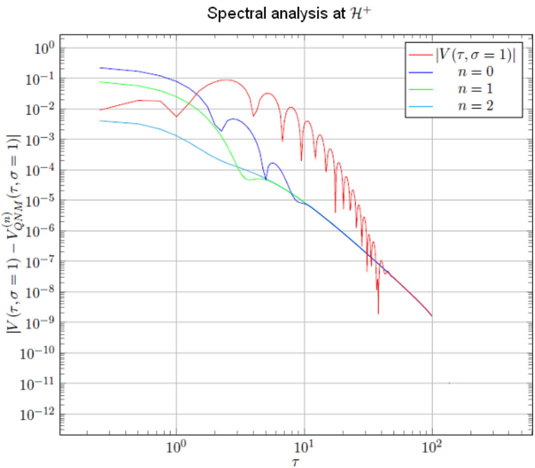
$$\hat{A}(s)\tilde{V} \overset{s \to s_k}{=} \hat{A}(s_k)\tilde{W}(\sigma) + \eta_k c = a + s_k b$$

$$c = \left(\frac{\partial \hat{A}}{\partial s}\right)(s_k) \cdot \bar{V}_k(\sigma)$$

- Solve $\hat{A}(s_k)\tilde{W}(\sigma) + \eta_k c = a + s_k b$ with spectral methods

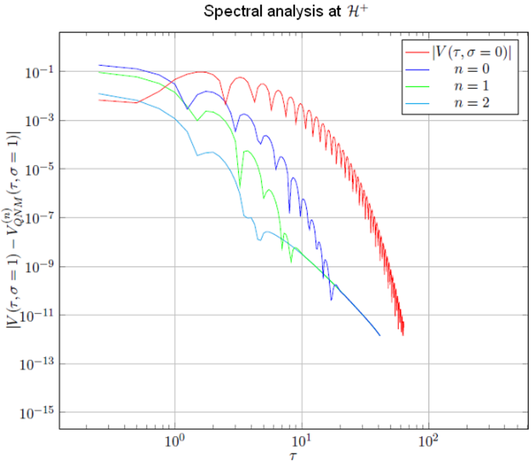


- Normalizing condition $\tilde{W}(1) =: \tilde{W}_1$ necessary for unique solution $\{\tilde{W}(\sigma), \eta_k\}$, because of $\hat{A}(s_k)\bar{V}_k = 0$
- Choice of normalization has no effect on result for η_k
- $\{V_0, \dot{V}_0\} \to V_{QNM}(\tau, \sigma) = \sum_{k=0}^{\infty} 2Re\{\eta_k \bar{V}_k(\sigma) e^{s_k \tau}\}$



Tail becomes visible at earlier times.

$$\lambda = 0$$
$$l = 1$$
$$V_0 = \sigma(1-\sigma)$$
$$\dot{V}_0 = 0$$



$$\lambda = 0$$
$$l = 2$$
$$V_0 = \sigma(1-\sigma)$$
$$\dot{V}_0 = 0$$

Hidden tail can be uncovered!

Summary

- Deformation of Bromwich integral hints at possible decomposition: $V(\tau,\sigma) = V_{QNM}(\tau,\sigma) + V_{tail}(\tau,\sigma)$
- Excitation coefficients η_k can be calculated from initial data $\{V_0(\sigma), \dot{V}_0(\sigma)\}$:

$$V_{QNM}(\tau,\sigma) = \sum_{k=0}^{\infty} 2Re\{\eta_k \bar{V}_k(\sigma) e^{s_k \tau}\}$$

- Construction of special initial data without quasi-normal ringing ($\eta_k = 0\ \forall k$) or tail ($\eta(s) \equiv 0$) possible.

Outlook

- Extension of QNM spectral analysis to Kerr-TE
- Tail excitation $\eta(s)$ from initial data
- Proof whether decomposition

$$V(\tau,\sigma) = V_{QNM}(\tau,\sigma) + V_{tail}(\tau,\sigma)$$

is (always?) possible

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